A TREATISE

ADJUSTMENT OF OBSERVATIONS,

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PREFACE.

In the following treatise I have endeavored to give a systematic account of the method of adjusting observations founded on the principle of the mean The more important applications, especially with reference to geodetic and astronomical work, are fully discussed

It has been my aim throughout to be practical The book originated and grew amid actual work, and hence subjects that are interesting mainly because they are curious, and methods of reduction that have become antiquated, are not noticed

Several of the views enunciated are not in the usual strain, but they are, however, such as I think all experienced observers, though perhaps not all mathematicians, will at once assent to.

As regards notation I have been conservative, usually following Encke's system as given by Chauvenet Some minor changes have been introduced which it is thought will tend to greater clearness and uniformity of expression

The examples and illustrations have been drawn chiefly from American sources, for the reason that much valuable material of this kind is to be had, and that thus far it has not been used for this purpose. They have been taken from records of work actually done, and principally from work with which I have been connected

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In the applications to practical work I have aimed at giving only so much of methods of observing as would serve to make the methods of adjustment intelligible It has been difficult to do this succinctly and at the same time satisfactorily, and accordingly references are given to books where descriptions of instruments and modes of using them can be found

Special attention has been given to the explanation of checks of computation, of approximate methods of adjustment, and of approximate methods of finding the precision of the adjusted values But in order to see how far it is allowable to use these short cuts the rigid methods must first be derived It is for this reason principally that the subject of triangulation has been dwelt on at such length In general it is unnecessary to spend a great amount of time in finding the probable error, when, after it has been found, it in many cases tells so little

I have been careful to give references to original authorities as far as I could ascertain them, and also to give lists of memoirs on special subjects which will be of use to any one desiring to follow those subjects farther Of recent writers I am indebted chiefly to Helmert and Zachariæ I desire also to acknowledge my obligations to my old Lake Survey friends, Messrs C C Brown, J H Darling, E S Wheeler, R S Woodward, and A Ziwet, who have read the manuscript in part and given me the benefit of their advice Mr Brown deserves special mention for assistance rendered while the book was passing through the press

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THE ADJUSTMENT OF OBSERVATIONS.

CHAPTER I

INTRODUCTION

THE factors that enter into the measurement of a quantity are, the observer, the instrument employed, and the conditions under which the measurement is made

1. **The Instrument.**—If the measure of a quantity is determined by untrained estimation only, the result is of little value The many external influences at work hinder the judgment from deciding correctly. For example, if we compare the descriptions of the path of a meteor as given by a number of people who saw the meteor and who try to tell what they saw, it would be found impossible to locate the path satisfactorily The work of the earlier astronomers was of this vague kind There was no way of testing assertions, and theories were consequently plentiful

The first great advance in the science of observation was in the introduction of instruments to aid the senses The instrument confined the attention of the observer to the point at issue and helped the judgment in arriving at conclusions As with a rude instrument different observers would get the same result, it is not to be wondered at that for a long time a single instrumental determination was considered sufficient to give the value of the quantity measured

The next advance was in the questioning of the instrument and in showing that a result better on the whole than a single direct measurement could be found This opened

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the way for better instruments and better methods of observation. For example, Gascoigne's introduction of cross-hairs into the focus of the telescope led to better graduated circles and to better methods of reading them, resulting finally in the reading microscopes now almost universally used   The culminating point was reached by Bessel, who, by his systematic and thorough investigation of instrumental corrections and methods of observation, may be said to have almost exhausted the subject   He confined himself, it is true, to astronomical and geodetic instruments, but his methods are of universal application

The questioning of an instrument naturally arises from noticing that there are discrepancies in repeated measurements of a magnitude with the same instrument or in measures made with different instruments   Thus, if a distance was measured with an ordinary chain, and then measured with a standard whose length had been very carefully determined, and the two measurements differed widely, we should suspect the chain to be in error and proceed to examine it before further measuring   So discrepancies found in measurements made with the same measure at different temperatures have shown the necessity of finding the length of the measure at some fixed temperature, and then applying a correction for the length at the temperature at which the measurement is made

Corrections to directly measured values are thus seen to be necessary, and to be due to both internal and external causes   The internal causes arising from the construction of the instrument are seen to be in great measure capable of elimination   From geometrical considerations the observer can tell the arrangement of parts demanded by a perfect instrument   He can compute the errors that would be introduced by certain supposed irregularities in form and changes of condition   The instrument-maker cannot, it is true, fulfil the conditions necessary for a perfect instrument, but he has been gradually approaching them more and more closely   It is to be remembered that, even if an

instrument could be made perfect at any instant, it would not remain so for any great length of time

It hence followed as the next great advance that the instrument was made adjustable in most of its parts, so that the relative positions of the parts are under the control of the observer This is getting to be more and more the case with the better class of instruments

Not only is error diminished by the improved construction of the instrument, but also by more refined methods of handling it It may be, indeed, that some contrivances beyond those required to make necessary readings for the measure of the quantity in question may be needed Thus, with a graduated circle regular or periodic errors of graduation may be expected If the angle between two signals were read with a theodolite, the reading on each signal, and consequent value of the angle, would be influenced by the periodic errors of the circle of the instrument Though a single vernier or microscope would suffice to read the circle when the telescope is directed to the signals, yet, as the circle is incapable of adjustment, we can only get rid of the influence of the periodicity by employing a number of verniers or microscopes placed at equal intervals around the circle It happens that this same addition of microscopes eliminates eccentricity of the graduated circle as well.

This same principle of making the method of observation eliminate the instrumental errors is carried through even after the nicest adjustments have been made Thus, in ordinary levelling, if the backsights and foresights are taken exactly equal the instrumental adjustment may be poor and still good work may be done But good work is more likely if the adjustments have been carefully made, as if for unequal sights, and still the sights are taken equal.

Simplicity of construction in an instrument is also to be aimed at An instrument that theoretically *ought* to work perfectly is often a great disappointment in practice. Two striking examples are the compensating base-apparatus and the repeating theodolite, both of which have been aban-

doned on all the leading surveys   In both cases the mechanical difficulties in the way have proved insurmountable, and the instruments have been replaced by others of simpler construction, to whose readings corrections can either be computed and applied or the errors of the readings can be eliminated by the method of observation   In this way no hopes of an accuracy which cannot be realized are held out

Such is the perfection now attained in the construction of mathematical instruments, and the skill with which they can be manipulated, that comparatively little trouble in making observations arises from the instrument itself

2 External Conditions.—The great obstacles to accurate work arise from the influence of external conditions—conditions wholly beyond the observer's or instrument-maker's control, and whose effect can, in general, neither be satisfactorily computed nor certainly eliminated by the method of observation   We have no means of finding the complex laws of their action   Many of them can be *avoided* by not observing while they operate in any marked degree   Thus, if while an observer was reading horizontal angles on a lofty station a strong wind should spring up, it would be useless for him to continue the work   If the air commenced to "boil" he should stop   If the sun shone on one side of his instrument its adjustments would be so disturbed that good work could not be expected   So in comparisons of standards   Comparisons made in a room subject to the temperature variations of the outside air would be of little value.   The standards should not only not be exposed to sudden temperature changes during comparisons, but at no other time, for it has been shown by recent experiments that the same standard may have different lengths at the same temperature after exposure to wide ranges of temperature *

The effects of external disturbances may sometimes be eliminated, in part at least, by the method of observation

* *American Journal of Science*, July, 1881

In the measurement of houzontal angles wheie the instrument is placed on a lofty station, the influence of the sun causes the centre post oi tripod of the station to twist in one direction during the day When this influence is removed at night the twist is in the opposite direction Assuming the twist to act uniformly, its effect on the results is eliminated by taking the mean of the readings on the signals observed in order of azimuth and then immediately in the reverse oider

Atmospheric refraction is another case in point In observing for time with an astronomical tiansit the effect of refraction on the mean of the recoided ieadings is eliminated by taking the star on the same number of threads on each side of the middle thread On the other hand, in the measurement of horizontal angles, if long lines aie sighted over, oi lines passing fiom land over large bodies of water or over a countiy much broken, the effects of refiaction are apt to be very maiked As we have no means of eliminating the discordances arising in this way by the method of obseivation, all we can do is, while planning a triangulation, to avoid as far as possible the introduction of such lines

It may happen that the effect of the exteinal disturbances on the observations can be computed approximately from theoretical considerations assuming a certain law of operation If the coirection itself is small this is allowable As an example take the zenith telescope, with which the method of observing for latitude is such that the correction foi iefraction is so small that the error of the computed value is not likely to exceed other eirois which entei into the work

3 **The Observer.**—Lastly we come to the observer himselt as the third element in making an observation Like the external conditions, he is a variable factor , all new observers certainly are

The observer, having put his instrument in adjustment and satisfied himself that the external conditions aie favorable, should not begin woik unless he considers that he himselt is in his normal condition If he is not in that condition

he introduces an unknown disturbing element unnecessa-
rily   He is also more liable to make mistakes in his read-
ings and in his record   For the same reason he should not
continue a series of observations too long at one time, as
from fatigue the latter part of his work will not compare
favorably with the first   In time-determinations, for in-
stance, nothing is gained by observing from dark until
daylight

The observer is supposed to have no bias   A good ob-
server, having taken all possible precautions with the ad-
justments of his instrument and knowing no reason for not
doing good work, will feel a certain amount of indifference
towards the results obtained   The man with a theory to
substantiate is rarely a good observer, unless, indeed, he
regards his theory as an enemy and not as a thing to be
fondled and petted

The greater an observer's experience the more do his
habits of observation become fixed, and the more mechanic-
al does he become in certain parts of his work   But his
judgment may be constantly at fault   Thus with the as-
tronomical transit he may estimate the time of a star cross-
ing a wire in the focus of the telescope invariably too soon
or invariably too late, according to the nature of his tem-
perament   If he is doing comparison work involving mi-
crometer bisections, he may consider the graduation mark
sighted at to be exactly between the centre wires of the
microscope when it is constantly on the same side of the
centre   This fixed peculiarity, which none but experienced
observers have, is known as their personal error

In combining one observer's results with those of another
observer we must either find by special experiment the dif-
ference of their personal errors and apply it as a correction
to the final result, or else eliminate it by the method of ob-
servation   Thus in longitude work the present practice
is to eliminate the effect of personal error from the final
result by having the observers change places at the middle
of the work

It is always safer to eliminate the correction by the
method of observing rather than by computing for it   For
though it may happen that so long as instruments and
conditions are the same the relative personal error of two
observers may be constant, yet some apparently trifling
change of conditions, such, for example, as illuminating
the wires of the instrument differently, may cause it to be
altogether changed in character

On account of personal error, if for no other reason, it
is evident that no number of sets of measures obtained in
the same way by a single observer ought to be taken as
furnishing a final determination of the value of a quantity
We must either vary the form of making the observations
or else increase the number of observers, in the hope that
personal error will eliminate itself in the final combination
of the measures

4 When all known corrections for instrument, for exter-
nal conditions, and for peculiarities of the observer have
been applied to a direct measure, have we obtained a cor-
rect value of the quantity measured ?   That we cannot say
If the observation is repeated a number of times with equal
care different results will in general be obtained

The reason why the different measures may be expected
to disagree with one another has been indicated in the pre-
ceding pages   There may have been no change in the con-
ditions of sufficient importance to have attracted the ob-
server's attention when making the observations, but he
may have handled his instrument differently, turned certain
screws with a more or less delicate touch, and the exter-
nal conditions may have been different   What the real dis-
turbing causes were he has no means of knowing fully   If
he had he could correct for them, and so bring the meas-
ures into accordance   Infinite knowledge alone could do
this   With our limited powers we must expect a residuum
of error in our best executed measures, and, instead of cer-
tainty in our results, look only for probability

The discrepancies from the true value due to these un-

explained disturbing causes we call *errors*   These errors
are *accidental,* being wholly beyond all our efforts to con-
trol   As soon as they are known to be *constant,* or we learn
the law of their operation, they cease to be classed as
errors

A very troublesome source of discrepancies in measured
values arises from *mistakes* made by the observer in reading
his instrument or in recording his readings   Mistakes from
imperfect hearing, from transposition of figures and from
writing one figure when another is intended, from mistak-
ing one figure on a graduated scale for another, as 7 for 9,
3 for 8, etc , are not uncommon   These also must be classed
as accidental errors, theoretically at least

Having, therefore, taken all possible precautions in mak-
ing the observations and applied all known corrections to
the observed values, the resulting values, which we shall in
future refer to as the *observed values,* may be assumed to
contain only accidental errors   We are, then, brought face
to face with the question, How shall the value of the quan-
tity sought be found from these different observed values ?

Synopsis of Mathematical Principles Employed

For convenience, and in order to avoid multiplicity of
references, the leading principles of pure mathematics made
use of in the further development of our subject are here
placed together

5 **Probability.**—(1) *The probability of the occurrence of
an event is represented by the fraction whose denominator is the
number of possible occurrences, all of which are supposed to be
independent of one another and equally likely to happen, and
whose numerator is the number of these occurrences favorable to
the event in question*   Thus if an event may happen in a ways
and fail in b ways, all equally likely to occur, the probability
of its happening is $\dfrac{a}{a+b}$, and of its failing $\dfrac{b}{a+b}$, certainty
being represented by unity

(2) *If there are n events independent of each other, and the probability of the first happening is* ψ_1, *of the second* ψ_2, *and so on, the probability that all will happen is*

$$\psi_1 \cdot \psi \qquad \cdot \psi_n$$

Thus if an urn contains two white and seven black balls the probability of drawing white at each of the first two trials, the ball not being replaced before the second trial, is

$$\frac{2}{9} \times \frac{1}{8} = \frac{1}{36}$$

6 Definite Integrals. — (a) To find the value of

$$\int_0^\infty e^{-t^2} dt$$

Let $t = xz$, then regarding x as a constant in integrating, we have

$$\int_0^\infty e^{-t^2} dt \doteq \int_0^\infty e^{-x^2 z^2} x \, dz$$

Multiply each member by $\int_0^\infty e^{-x^2} dx$, which is equal to $\int_0^\infty e^{-t^2} dt$, since the limits of integration are the same; then

$$\left\{ \int_0^\infty e^{-t^2} dt \right\}^2 = \int_0^\infty dz \int_0^\infty e^{-x^2(1+z^2)} x \, dx$$

$$= \int_0^\infty \frac{dz}{2(1+z^2)!}$$

$$= \frac{\pi}{4}$$

Hence $\int_0^\infty e^{-t^2} dt = \frac{1}{2} \sqrt{\pi}$

(b) To find the value of $\int_{-\infty}^\infty at^2 e^{-a^2 t^2} dt$:

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Integrating by parts, we have

$$\int_{-\infty}^{\infty} at^2 e^{-a^2 t^2} dt = \left(-\frac{t e^{-a^2 t^2}}{2a}\right)_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-a^2 t^2}}{2a^2} d(at)$$

$$= \frac{\sqrt{\pi}}{2a^2}$$

Also, similarly,

$$\int_{-\infty}^{\infty} t^4 e^{-a^2 t^2} dt = \frac{3\sqrt{\pi}}{4a^5}$$

(c) To find the value of $\int_0^a e^{-t^2} dt$:

The value of this integral cannot be expressed exactly in a finite form, but may be found approximately as follows

Expanding e^{-t^2} in a series and integrating each term separately, we have

$$\int_0^a e^{-t^2} dt = \int_0^a \left(1 - \frac{t^2}{1} + \frac{t^4}{1\,2} - \quad \cdot\right) dt$$

$$= a - \frac{a^3}{3} + \frac{1}{1\,2}\frac{a^5}{5} - \quad \cdot$$

This series is convergent for all values of a, but the convergence is only rapid enough for small values of a

For large values of a it is better to proceed as follows·
Integrating by parts,

$$\int e^{-t^2} dt = \int -\frac{1}{2t} de^{-t^2}$$

$$= -\frac{1}{2t} e^{-t^2} - \frac{1}{2} \int \frac{e^{-t^2}}{t^2} dt$$

$$= -\frac{1}{2t} e^{-t^2} + \frac{1}{2^2 t^3} e^{-t^2} + \frac{1\,3}{2^2} \int \frac{e^{-t^2}}{t^4} dt$$

Hence

$$\int_a^{\infty} e^{-t^2} dt = \frac{e^{-a^2}}{2a} \left\{ 1 - \frac{1}{2a^2} + \frac{1\,3}{(2a^2)^2} - \frac{1\,3\,5}{(2a^2)^3} + \quad \cdot \quad \right\}$$

But

$$\int_0^a e^{-t^2}dt = \int_0^\infty e^{-t^2}dt - \int_a^\infty e^{-t^2}dt$$

$$= \frac{\sqrt{\pi}}{2} - \int_a^\infty e^{-t^2}dt$$

finally,

$$\int_0^a e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} - \frac{e^{-a^2}}{2a}\left\{1 - \frac{1}{2a^2} + \frac{1\,3}{(2a^2)^2} - \frac{1\,3\,5}{(2a^2)^3} + \qquad \right\}$$

It is easily shown that, by stopping the summation at any term, the result will differ from the true value by less than the term stopped at

Approximate values of the expression $\int_0^a e^{-t^2}dt$ may be computed from the above formulas for any numerical value of a

7 Taylor's Theorem.—(a) If $f(x)$ is any function of x, and $f(x+h)$ is to be developed in ascending powers of h, then

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{1\,2}f''(x) + \qquad (1)$$

A more rapid approximation is obtained by putting the development in the following form

$$f\left(x + \frac{h}{2} + \frac{h}{2}\right) = f\left(x + \frac{h}{2}\right) + \frac{h}{2}f'\left(x + \frac{h}{2}\right)$$
$$+ \frac{1}{1\,2}\left(\frac{h}{2}\right)^2 f''\left(x + \frac{h}{2}\right) + \quad\cdot$$

$$f\left(x + \frac{h}{2} - \frac{h}{2}\right) = f\left(x + \frac{h}{2}\right) - \frac{h}{2}f'\left(x + \frac{h}{2}\right)$$
$$+ \frac{1}{1\,2}\left(\frac{h}{2}\right)^2 f''\left(x + \frac{h}{2}\right) -$$

By subtraction and transposition,

$$f(x+h) = f(x) + h\,f'\left(x + \frac{h}{2}\right) + \frac{h^3}{24}f'''\left(x + \frac{h}{2}\right) + \quad\cdot\cdot(2)$$

(b) Let F denote a function of a series of quantities X, Y,   expressed by the relation

$$F = f(X, Y, \quad)$$

and let X', Y',   denote approximate values of X, Y, and x, y,   the corrections to these approximate values, so that

$$X = X' + x$$
$$Y = Y' + y$$

then

$$F = f(X', Y', \quad . \quad) + \frac{\delta F}{\delta X'}\, x + \frac{\delta F}{\delta Y'}\, y$$
$$+ \frac{\delta^2 F}{\delta X'^2} x^2 + 2\frac{\delta^2 F}{\delta X' \delta Y'}\, xy + \frac{\delta^2 F}{\delta Y'^2}\, y^2$$
$$+$$

where $\dfrac{\delta F}{\delta X'}$, $\dfrac{\delta F}{\delta Y'}$,   . are the values found by differentiating $f(X, Y, \quad . \quad)$, . with respect to X, Y,   and then substituting X', Y',   . for X, Y,   .

If the corrections x, y,   are so small that their squares and higher powers may be neglected and they are written dX', dY',   and $F - f(X', Y', \quad . \quad)$ is written dF, then

$$dF = \frac{\delta F}{\delta X'}\, dX' + \frac{\delta F}{\delta Y'}\, dY' + \quad .$$

which is exactly the result found by differentiating F, which is a function of X, Y,   with respect to these quantities

We shall use one form or the other as may be most convenient.

Ex 1 If v is a very small correction to the number N, required to express $\log(N + v)$ in the linear form

We have

$$\log (N+v) = \log N + \frac{d}{dN}(\log N)\ v + \quad.$$

$$= \log N + \frac{\text{mod}}{N}\ v +$$

where mod is the modulus of the common system of logarithms

With a seven-place table we may use the formula

$$\log (N+v) = \log N + \delta_N\, v$$

where δ_N is the tabular difference corresponding to one unit for the number

For small numbers however, it is better to take δ_N from the table for the form

$$\tfrac{1}{2}\left\{\log (N+1) - \log (N-1)\right\}$$

Thus from the table

$$\log (6543\ 2 + v) = 3\ 8157902 + 0\ 0000664\ v$$
$$\log (654\ 32 + v) = 2\ 8157902 + 0\ 0006637\ v$$
$$\log (65\ 432 + v) = 1\ 8157902 + 0\ 0066378\ v$$

Ex 2 In a ten-place log table where angles are given at regular intervals, required to find log sin $(A + a)$ when A is given in the table and a is a number of seconds less than the tabular interval

We have

$$\log \sin (A + a) = \log \sin A + \frac{d}{dA}\left\{\log \sin \left(A + \frac{a}{2}\right)\right\} a$$

$$= \log \sin A + \text{mod}\ \sin 1''\ a\ \cot \left(A + \frac{a}{2}\right)$$

Now,

$$\log \text{mod} = 9\ 6377843 - 10$$
$$\log \sin 1'' = 4\ 6855749 - 10$$
$$\log 10^7 = 7$$

$$\overline{1\ 3233592}$$

which expresses log (mod sin 1") in terms of the seventh place of decimals as the unit Hence

$$\log \sin (A + a)$$
$$= \log \sin A + \log^{-1}\left\{1\ 3233592 + \log a + \left(\log \cot A + \frac{a}{2} \times \text{diff for } 1''\right)\right\}$$

With Vega's Thesaurus,* which gives the log functions to single seconds to the end of the first degree and afterwards for every 10", log sin $(A + a)$ can be found from the above expression for values of $A > 3°$, and also when A lies between 20' and 2° to within less than unity in the tenth decimal place Between 2° and 3° the difference may be as large as 3 units in the tenth place from the value found by carrying out the formula more exactly But in the

* *Thesaurus Logarithmorum Completus* Lipsiæ, 1794

Thesaurus, in the trigonometrical part, "the uncertainty of the last figure amounts to 4 units '* Hence the above process is in general sufficient when this table is used

With a seven-place table of log sines it would be, in general, sufficient to take the tabular difference δ_A for $1''$ for the angle A as the value of

$$\frac{d}{dA} \left\{ \log \sin \left(A + \frac{a}{2} \right) \right\}$$

so that

$$\log \sin (A + a) = \log \sin A + \delta_A \, a$$

Ex 3 If A is the approximate value of an angle, and v a correction to it so small that its square and higher powers may be neglected, required to express $\log \sin (A + v)$ in the linear form, using a ten-place table

Let A_1 be the angle nearest to A in the table, and set

then
$$A = A_1 + a$$

$\log \sin (A + v)$
$= \log \sin (A_1 + a + v)$

$= \log \sin A_1 + \text{mod} \, \sin 1'' \, a \cot \left(A_1 + \frac{a}{2} \right) + \text{mod} \, \sin 1'' \cot (A_1 + a) \, v$

$= \log \sin A_1 + \log^{-1} \left\{ 1\ 3233592 + \log a + (\log \cot A_1 + \frac{a}{2} \times \text{diff for } 1'') \right\}$

$\qquad + \log^{-1} \left\{ 1\ 3233592 + (\log \cot A_1 + a \times \text{diff for } 1'') \right\} \, v$

Expand $\log \sin (68°\ 16'\ 32''\ 076 + v)$

$$A_1 = 68°\ 16'\ 30''$$
$$a = \qquad 2''\ 076$$

$$\begin{array}{ll} & 1\ 3233592 \\ \log \cot A_1 & 9\ 6003780 \\ \hline & 0\ 9237372 \\ a \times \text{diff } 1'' & -127 \\ \hline & 0\ 9237245 \qquad 8,3893 \\ -\frac{a}{2} \times \text{diff } 1'' & +64 \\ \hline & 0\ 9237309 \\ \log a & 0\ 3172273 \\ \hline & 1\ 2409582 \end{array}$$

$\log \sin 68°\ 16'\ 30'' \qquad\qquad\qquad 17,416$
$$9\ 9680022,271$$

Hence $\log \sin (68°\ 16'\ 32''\ 076 + v) = \qquad 9\ 9680039,687 + 8,3893 \, v$

when the difference is expressed in terms of the seventh decimal place as the unit

* Bremiker's edition of Vega, translated by Fischer   Preface, p 10

With a seven-place table, except for small angles or angles near 180° it will be sufficient to take

$$\log \sin (A_1 + a + v) = \log \sin A_1 + \delta_A a + \delta_A v$$

when δ_A is the tabular difference corresponding to $1''$ for the angle A_1 It can be taken by inspection from the table Thus,

$$\log \sin (68°\ 16'\ 32'' + v) = 9\ 9680039 + 8,4\ v$$

8 Interpolation.—So far as interpolation is concerned, we have mainly to deal with the logarithms of trigonometric functions   The differences between the successive values given in a table are *first differences*, and the differences between the successive first differences are *second differences* Beyond second differences we do not need to go

This may be expressed in tabular form

Function	First diff	Second diff
$f(A)$		
	d_1	
$f(A+a)$		d_2
	d_1'	
$f(A+2a)$		

Hence

$$f(A+a) = f(A) + d_1$$
$$f(A+2a) = f(A+a) + d_1'$$
$$= f(A) + 2d_1 + d_2$$

Generally

$$f(A+na) = f(A) + nd_1 + \frac{n(n-1)}{2}\ d_2 + \cdots$$

which is Newton's formula

Ex In a ten-place table of log sines in which values are

given to every 10 seconds, required $\log \sin A$ when A is any angle

Let $A_1 =$ the part of the given angle A to the nearest 10 seconds that occurs in the table

$a =$ the units of seconds and parts of seconds in the given angle

$d_1, d_2 =$ the first and second tabular differences

then

$$\log \sin A = \log \sin \left(A_1 + \frac{a}{10}\ 10 \right)$$

$$= \log \sin A_1 + \frac{a}{10} d_1 + \tfrac{1}{2} \frac{a}{10} \left(\frac{a}{10} - 1 \right) (- d_2)$$

$$= \log \sin A_1 + a \frac{d_1}{10} + \tfrac{1}{2} a (10 - a) \frac{d_2}{100}. \qquad (1)$$

Writing this in the form,

$$\log \sin A = \log \sin A_1 + a \left\{ \frac{d_1}{10} + \left(5 - \frac{a}{2} \right) \frac{d_2}{100} \right\} \quad (2)$$

we have the convenient rule   Assume the second difference constant throughout the interval a Then from the first and second differences find by simple interpolation the value of the difference at the middle of the interval   This difference multiplied by the interval gives the correction to the tabular log sine

To find $\log \sin 68°$ $16'$ $32''$ 076

From the table,

$\log \sin 68°$ $16'$ $30'' =$ $9\ 9680022,271$
$\qquad\qquad\qquad\quad d_1 = 83,889$ for $10''$ in units of the seventh decimal place
$\qquad\qquad\qquad\quad d_2 =\ 0,012$

Hence, from equation (1),

Corr to tab value $= 2\ 076 \times 8,3889 + \dfrac{2\ 076}{2} (10 - 2\ 076) \dfrac{12}{100}$

$\qquad\qquad\qquad = \qquad 17,416$

$\log \sin 68°$ $16'$ $32''$ $076 = 9\ 9680039,687$

The difference at $35'' = 8,3889$
We want it at $31'' 038$, the middle of the interval
Now, change of first difference $= 0,00012$ for $1''$
Hence corr to first difference $= (35 - 31\ 038) \times 0,00012$

$$= 0,0005$$

First diff $= 8,3889$

Diff required $= \overline{8,3894}$

And $2\ 076 \times 8,3894 = 17,416$ as before

9 Periodic Series.—To sum the series

$$\text{Cos } o + \cos \theta + \cos 2\theta + \qquad + \cos (n-1)\theta$$
$$\text{Sin } o + \sin \theta + \sin 2\theta + \qquad + \sin (n-1)\theta$$

where $\theta = \dfrac{360}{n}$, n being an integer, we may proceed as follows

If θ is the angle which a line $B'OB$ makes with OA then the projection of OB on OA is $OB \cos \theta$, and the projection of OB' is $-OB \cos \theta$ if $OB' = OB$ The projections on a line at right angles to OA are $OB \sin \theta$ and $-OB \sin \theta$ respectively

If we divide the circumference of a circle into n equal parts at the points $A, B,$ then each angle at the centre O is $\dfrac{360°}{n}$, or θ, and by projecting the lines $OA, OB,$ on the diameter through A we find the sum of the first series to be zero, and by projecting the same lines on the diameter perpendicular to OA we find the sum of the second series to be also zero

These results may be written

$$\Sigma \cos m\theta = 0 \qquad\qquad \Sigma \sin m\theta = 0$$

where m assumes all values from o to $n - 1$

Hence it follows that

$$\Sigma \sin m\theta \, \cos m\theta = \tfrac{1}{2} \, \Sigma \sin m\, 2\theta = 0$$

$$\Sigma \cos^2 m\theta \qquad = \frac{n}{2} + \tfrac{1}{2}\Sigma \cos m\, 2\theta = \frac{n}{2}$$

$$\Sigma \sin^2 m\theta \qquad = \frac{n}{2} - \tfrac{1}{2}\Sigma \cos m\, 2\theta = \frac{n}{2}$$

5

10 Notation.—The following convenient notation, intro-
duced by Gauss, is now very generally used in the method
of least squares

If a_1, a_2, . are quantities of the same kind, their alge-
braic sum is denoted by $[a]$, and the sum of their squares by
$[aa]$ or $[a^2]$, so that

$$[a] = a_1 + a_2 + \quad . \quad + a_n$$
$$[aa] \text{ or } [a^2] = a_1^2 + a_2^2 + \quad \quad + a_n^2$$

Also,

$$[ab] = a_1 b_1 + a_2 b_2 + \quad \quad + a_n b_n$$
$$\left[\frac{ab}{c}\right] = \frac{a_1 b_1}{c_1} + \frac{a_2 b_2}{c_2} + \quad \quad + \frac{a_n b_n}{c_n}$$

We shall use the symbol $[a$ to denote the sum of a series
of quantities a all taken with the same sign

CHAPTER II

THE LAW OF ERROR

The Arithmetic Mean

11 (*a*) **When the quantity measured is the quantity to be found.**—If M_1, M_2 — M_n are n direct and independent measures of a quantity, T, we may write

$$\left.\begin{array}{l} T-M_1=\varDelta_1 \\ T-M_2=\varDelta_2 \\ \quad\cdot\qquad\cdot \\ T-M_n=\varDelta_n \end{array}\right\} \qquad (1)$$

where $\varDelta_1$, $\varDelta_2$, $\varDelta_n$ indicate the differences between T and the observed values, and are therefore the errors of observation

We have here n equations and $n + 1$ unknowns   What principle shall we call to our aid to solve these equations and so find T, $\varDelta_1$, $\varDelta_2$, $\varDelta_n$? In answering this question I shall follow the order of natural development of the subject, which, in the main, is also the order of its historical development

The value sought must be some function of the observed values and fall between the largest and smallest of them   If the observed values are arranged according to their magnitudes they will be found to cluster around a central value   On first thoughts the value that would be chosen as the value of T would be the central value in this arrangement if the number of observations were odd, and either of the two central values if the number were even   In other words, a plausible value of the unknown would be that observed value which had as many observed values greater than it as it had less than it   Now, since a small change in any of the observed values, other than the central

value, would in general produce no change in the result, the number of observations remaining the same, this method of proceeding might be regarded as giving a plausible result, more especially if the observed values were widely discrepant

On the other hand, the taking of the central value is objectionable, because it gives the preference to a single one of the observed values, while if these values are supposed to be equally worthy of confidence, as it is reasonable to take them in the absence of all knowledge to the contrary, each ought to exert an equal influence on the result   We may, therefore, with more reason assume the value of T to be a *symmetrical function* of the observed values *   Also, since a change in the number of observations should produce no change in the form of this function, it follows that the function must be of such a form as to satisfy the condition that when the observed values are all equal to one another it will reduce to this common value, that is, if $T = f(M_1, M_2 \quad M_n)$, and $M_1 = M_2 = \quad M_n = M$, then $f(M, M, \quad . M) = M$ For if we had a single observation, then necessarily $f(M) = M$

Let, then, V be a symmetrical function of $M_1, M_2 \quad M_n$, and put

$$V - M_1 = v_1$$
$$V - M_2 = v_2 \qquad\qquad (2)$$

$$V - M_n = v_n$$

Expanding by Taylor's theorem

$$V = f(M_1, M_2 \quad M_n)$$
$$= f(V - v_1, V - v_2, \quad)$$
$$= f(V, V, \quad) - [v]\frac{df}{dv} + H + K + \qquad (3)$$

where $H, K, \quad$ are terms involving the second, third, . powers of the small quantities, $v_1, v_2, \quad v_n$

* See Reuschle, *Crelle Jour Math* , vol xxvi , Schiaparelli, *Rendiconti del R Instituto Lombardo*, 1868 , *Astron Nachr* , 2068, 2097 Stone, *Month Not Roy Astron Soc* , vol xxxiii , *Astron Nachr* , 2092 , Ferrero, *Expos del Met dei Min Quadr* , Florence, 1876   Also Fechner, *Ueber den Ausgangswerth der kl Abweichungssumme*, Leipzig, 1874

The simplest symmetrical function of the observed values that can be chosen as the form for V is their arithmetic mean—that is, $\frac{[M]}{n}$ If we take V equal to this value, then from equation (1) by addition $[v]=0$, and, neglecting powers of v higher than the first, equation (3) is satisfied identically Thus far, therefore, it would appear that the arithmetic mean may be taken as one solution   It may happen that the values $M_1, M_2, \quad M_n$ are of such a nature that some other symmetrical function than the arithmetic mean will satisfy (2) better than will the arithmetic mean   That the arithmetic mean is *on the whole* the best form for the function $f(M_1, M_2, \quad M_n)$, when $M_1, M_2, \quad M_n$ are direct measures of some phenomenon in the sciences of observation, which sciences only we intend to consider, may be confirmed by a comparison of results flowing from this hypothesis with the records of experience   This we shall do later (see Art 24, 37, 51)

The older mathematicians, as Cotes and Simpson, laid the foundations of our subject by announcing the principle of the arithmetic mean   Gauss, to whom we owe the first complete exposition, assumed the arithmetic mean as a plausible hypothesis,* and Hansen, who made the next great advances, started from it as an axiom   The principle itself may be stated as follows   If we have n observed values of an unknown, all equally good so far as we know, the most plausible value of the unknown (best value on the whole) is the arithmetic mean of the observed values

12 By adding equations (1), Art 11, and taking the mean, we have

$$T = \frac{[M]}{n} + \frac{[\Delta]}{n}$$

$$= V + \frac{[\Delta]}{n}$$

The last term of this equation will become very small if,

* Gauss' words are, "Axiomatis loco haben solet hypothesis" (*Theoria Motus*, lib 2, sec 3)

n being very large, the sum $[\varDelta]$ of the errors remains small   Now, if, after making one observation and before making another, we readjust our instrument, determine anew its corrections, choose the most favorable conditions for observing, and vary the form of procedure as much as possible, it is reasonable to suppose that the disturbing influences will balance one another in the result following from the proper combination of the observed values   It may take an infinite number of trials to bring this about   In the absence of all knowledge we cannot say that it will take less   And, reckoning mistakes in reading the instrument or in recording the readings as accidental errors, an infinity of a higher order than the first may be required to eliminate them

In other words, there being no reason to suppose that an error in excess (or positive error) is more likely to occur on the whole than an error in defect (or negative error), we may, when n is a very large number, consider $\dfrac{[\varDelta]}{n}$ to be an infinitesimal with respect to T We may, therefore, in this case put

$$V = T$$

—that is, *when the number of observed values is very great the arithmetic mean is the true value*

13   From the principle of the arithmetic mean two important inferences may be derived   For, taking the arithmetic mean, V, of n observed values of an unknown as the most plausible value of that unknown, we may write our *observation equations* in the form,

$$\left.\begin{array}{l} V - M_1 = v_1 \\ V - M_2 = v_{_2} \\ \\ V - M_n = v_n \end{array}\right\} \qquad (1)$$

where $v_1, v_{_2}, \ \cdot \ v_n$ are called the *residual errors* of observation, or simply the *residuals*

(a) By addition
$$nV - [M] = [v]$$
and
$$[v] = 0 \qquad (2)$$
—that is, *the sum of the residuals is zero*, in other words, *the sum of the positive residuals is equal to the sum of the negative residuals*

There is a very marked correspondence between the series in which n is infinitely great and T is the *true* value, and a series in which n is finite and the arithmetic mean V is taken as the best value attainable   For in the first case the sum of the *errors*, Δ, divided by n, is zero, and in the second the sum of the *residuals*, v, is zero

(b) Let V' be any assumed value of the unknown other than the arithmetic mean, and put
$$\left. \begin{array}{c} V' - M_1 = v'_1 \\ V' - M_2 = v'_2 \\ \\ V' - M_n = v'_n \end{array} \right\} \qquad (3)$$

From equations (1) and (3), by squaring and adding,
$$[vv] = nVV - 2V[M] + [MM]$$
$$[v'v'] = nV'V' - 2V'[M] + [MM]$$

Hence by a simple reduction,
$$[v'v'] = [vv] + n\left(V' - \frac{[M]}{n}\right)^2$$

Now, $\left(V' - \dfrac{[M]}{n}\right)^2$, being a complete square, is always positive
$$[v'v'] > [vv]$$
—that is, *the sum of the squares of the residuals v, found by taking the arithmetic mean, is a minimum*

Hence the name *Method of Least Squares*, which was first given by Legendre

Let us recapitulate the three forms of solution proposed

for finding the most plausible value V of the unknown from the n equations

$$V - M_1 = v_1$$
$$V - M_2 = v,$$

$$V - M_n = v_n$$

(1) Find all possible values of V and take the mean   The values of V are $M_1, M_2,$ M_n, since for each observation considered singly the best value must be the directly observed one, and the mean of these values is $\dfrac{[M]}{n}$

(2) Solve simultaneously, making $[v] = 0$. This gives

$$nV - [M] = [v] = 0$$

$$\text{or} \qquad V = \frac{[M]}{n}$$

the arithmetic mean

(3) Solve simultaneously, making $[vv] = $ a minimum. This gives

$$(V - M_1)^2 + (V - M_2)^2 + \; .. \quad (V - M_n)^2 = \text{a min}$$

Differentiating with respect to V,

$$V - M_1 + V - M_2 + \qquad . \; V - M_n = 0$$

$$\text{or} \qquad V = \frac{[M]}{n}$$

the arithmetic mean

14. **When the quantity measured is a function of the quantities to be found.**—We pass now to the more general but equally common case in which the observations, instead of being made directly on the quantity to be determined, are made indirectly—that is, made on a quantity which is a function of the quantities whose values are to be found

Thus, let the function connecting the observed quantity T and the unknowns X, Y, . be

$$T = f(X, Y, \quad .)\qquad\qquad(1)$$

in which the constants involved are given by theory for each observation

If M_1, M_2, are the observed values of T, the equations for finding the unknowns, when reduced to the linear form, may be written

$$\left.\begin{array}{l} v_1 + M_1 = a_1 X + b_1 Y + \quad . . - L_1 \\ v_2 + M_2 = a_2 X + b_2 Y + \qquad - L_2 \\ \qquad . \qquad . \qquad . \qquad . \end{array}\right\}\qquad(2)$$

in which a_1, b_1, L_1, . are known constants, and v_1, v_2, are the residual errors of observation

Now, if the number of equations, n, is equal to the number of unknowns, n_2, the values of X, Y, may be found by the ordinary algebraic methods, and if substituted in the equations will satisfy them exactly. But if the number of equations exceeds the number of unknowns, the values found from a sufficient number of the equations will not in general satisfy the remaining equations exactly. Many such sets of values may be found, which are therefore all *possible* solutions But of all these possible sets some one will satisfy the equations better than any of the others We have so far no means of knowing when we have found this *most plausible* (best on the whole) set of values With a single unknown the arithmetic mean gives the most plausible result Let us see if a method of finding means corresponding to those of Art 13 will apply to these equations

For simplicity in writing take the three equations .

$$a_1 X + b_1 Y = M_1 + v_1$$
$$a_2 X + b_2 Y = M_2 + v_2$$
$$a_3 X + b_3 Y = M_3 + v_3$$

where a_1, b_1, . . are known, and X, Y are to be found.

6

(1) Find all possible values of X and Y, and combine them

To do this we form all possible sets of two equations and solve each set   Thus,

$$a_1X+b_1Y=M_1 \qquad a_1X+b_1Y=M_1 \qquad a_2X+b_2Y=M_2$$
$$a_2X+b_2Y=M_2 \qquad a_3X+b_3Y=M_3 \qquad a_3X+b_3Y=M_3$$

whence at once

$$(a_1b_2-a_2b_1)X=b_2M_1-b_1M_2$$
$$(a_1b_2-a_2b_1)Y=a_1M_2-a_2M_1$$

$$(a_1b_3-a_3b_1)X=b_3M_1-b_1M_3$$
$$(a_1b_3-a_3b_1)Y=a_1M_3-a_3M_1$$

$$(a_2b_3-a_3b_2)X=b_3M_2-b_2M_3$$
$$(a_2b_3-a_3b_2)Y=a_2M_3-a_3M_2$$

In combining these values of X and of Y we are met by a difficulty   It would not do to take the arithmetic means as the most plausible values, for X and Y may be better determined from one set of equations than from another, and the arithmetic mean gives the most plausible value only on the assumption that all of the values combined in it are of equal quality   It is necessary, therefore, to have a method of combining observations of different quality before we can find X and Y in this way

(2) The simultaneous solution of the equations by making the sum of the residuals equal to zero.

Hence X, Y should be found from

$$a_1X+b_1Y-M_1+a_2X+b_2Y-M_2+a_3X+b_3Y-M_3=0$$

which is impossible, as this equation may be satisfied by an infinite number of values of X and Y The principle is, therefore, insufficient

(3) The simultaneous solution by making the sum of the squares of the residuals a minimum.   We have

$$(a_1X+b_1Y-M_1)^2+ \quad . \quad + \quad . \quad =\text{a min.}$$

Differentiating with respect to X, Y as independent variables, we find

$$[aa] X + [ab] Y = [aM]$$
$$[ab] X + [bb] Y = [bM]$$

where

$$[aa] = a_1 a_1 + a_2 a_2 + a_3 a_3$$
$$[ab] = a_1 b_1 + a_2 b_2 + a_3 b_3$$

This method gives as many equations as unknowns, and so but one set of values of X and Y can be found.

We cannot, however, say that we have found the most plausible values of X and Y. All we can say is that the last method employed reduces the number of equations to the number of unknowns and gives us one set of values of X and Y, and that the same principle applied to the special case of one unknown gives the most plausible value of that unknown, in that it gives the arithmetic mean Analogy, however, would lead us to suspect that we have found the most plausible values of X and Y. With one unknown, if the separate observed values, represented by lines of the proper length, are plotted along a straight line from a certain assumed origin, and equal weights are placed at the end points, the position of the centre of gravity of the weights will coincide with the end of the line representing the arithmetic mean of the distances, and the sum of the squares of the distances of the weights from the centre of gravity is a minimum.

Again, the centre of gravity of n equally well-observed positions of a point in space would be the most plausible mean position to take for the point But it is a well known principle that, if equal weights be placed at n points in space, the centre of gravity of these weights is at a point such that the sum of the squares of its distances from the weights is a minimum * On this principle Legendre founded the rule of minimum squares, and he employed the rule

* See, for example, Todhunter's *Statics*, Art 138

as giving a *convenient* method of solution in the class of problems under consideration

The Law of Error of a Single Observed Quantity

15   With a single unknown we have seen that the most plausible value is the arithmetic mean of the independently observed values, and that it can be found by making the sum of the squares of the residuals a minimum   The methods are equivalent

With more than one unknown we have failed to find this correspondence of methods   The reasoning from analogy in the preceding article is well enough as far as it goes, but it is not conclusive   The difficulty lies in combining values not equally good   We must, therefore, devise some method of combining such values before a rule for finding means can be applied to several unknowns.

Now, when several independent measures of the same quantity, all equally good, have been made, it must be granted that errors in excess and errors in defect are equally likely to occur to the same amount—that is, are equally probable   Experience shows that in any well-made series of observations small errors are likely to occur more frequently than large ones, and that there is a limit to the magnitude of the error to be expected   If, therefore, a denotes this limit or maximum error, we must consider all the errors of the series to be ranged between $+a$ and $-a$, but to be most numerous in the neighborhood of zero   Hence the probability of the occurrence of an error may be assumed to be a certain function of the error

If, then, the probability that an error does not exceed Δ be denoted by $\xi(\Delta)$, the probability of an error between Δ and $\Delta + d\Delta$ is

$$\xi(\Delta + d\Delta) - \xi(\Delta) = \xi'(\Delta)\, d\Delta$$
$$= \varphi(\Delta)\, d\Delta \quad \text{suppose} \qquad (1)$$

The function $\varphi(\Delta)$ is called the law of distribution of error, or simply the *law of error*.

The probability that an error falls between any assigned limits b and a is the sum of the probabilities $\varphi(\varDelta)\, d\varDelta$ extending from b to a, and is expressed in the ordinary notation of the integral calculus by

$$\int_a^b \varphi(\varDelta)\, d\varDelta \qquad (2)$$

Hence it follows that the probability that an error does not exceed the value a is

$$\int_{-a}^{+a} \varphi(\varDelta)\, d\varDelta \qquad (3)$$

The properties of errors stated above are not sufficient to determine the form of the function $\varphi(\varDelta)$ definitely Among other forms that might be chosen to satisfy them the simplest would be

$$\varphi(\varDelta) = e^{-h\Delta} \qquad (4)$$

where h is a constant

This was, indeed, that selected by Laplace in his investigation *Determiner le milieu que l'on doit prendre entre trois observations données d'un même phénomène* (Mém. Acad Paris, 1774) From this form may be readily derived the results stated in Art 11

The form of $\varphi(\varDelta)$ may, however, be more satisfactorily determined by calling in the aid of the calculus of probabilities   For if $\varphi(\varDelta_1)\, d\varDelta_1$, $\varphi(\varDelta_2)\, d\varDelta_2$,   denote the probabilities of the occurrence of errors in n observations between $\varDelta_1$ and $\varDelta_1 + d\varDelta_1$, $\varDelta_2$ and $\varDelta_2 + d\varDelta_2$,   .   respectively, the probability of the simultaneous occurrence of this system of errors is proportional to the product (see Art 5)

$$\varphi(\varDelta_1)\, \varphi(\varDelta_2) \qquad \varphi(\varDelta_n)$$

Denote this expression by ψ, so that

$$\psi = \varphi(\varDelta_1)\, \varphi(\varDelta_2) \qquad \varphi(\varDelta_n) \qquad (5)$$

Now, the true value T, and therefore the values of $\varDelta_1$, $\varDelta_2$, . $\varDelta_n$, are unknown   If we make the expression for ψ a

maximum, we should find the most probable value of the unknown But we have seen that the most plausible value of the unknown is the arithmetic mean of the observed values, and that when the number of observations is very large the arithmetic mean is the true value T Calling, then, the most plausible value the most probable value, we have, when n is large, the true value by making ψ a maximum The form of the function φ will, therefore, result from this hypothesis

Now, since $\log \psi$ varies as ψ, we must have $\log \psi$ a maximum, and therefore by differentiation

$$0 = \frac{d \log \psi}{dT} = \frac{\delta \log \varphi(\Delta_1)}{\delta \Delta_1} \frac{d\Delta_1}{dT} + \frac{\delta \log \varphi(\Delta_2)}{\delta \Delta_2} \frac{d\Delta_2}{dT} +$$

or

$$0 = \frac{1}{\psi} \frac{d\psi}{dT} = \Delta_1 \frac{\delta \log \varphi(\Delta_1)}{\Delta_1 \delta \Delta_1} + \Delta_2 \frac{\delta \log \varphi(\Delta_2)}{\Delta_2 \delta \Delta_2} + \qquad (6)$$

since from equation (1), Art 11,

$$\frac{d\Delta_1}{dT} = \frac{d\Delta_2}{dT} = \cdot \qquad = \frac{d\Delta_n}{dT} = 1$$

But, from the principle of the arithmetic mean, when the number of observations is very great,

$$\Delta_1 + \Delta_2 + \qquad + \Delta_n = 0. \qquad (7)$$

Also, since equations (6) and (7) must be simultaneously satisfied by the same value of the unknown, being the most probable value in either case, and since the errors Δ_1, Δ_2, Δ_n are connected only by the relation $[\Delta] = 0$, we necessarily have, when $n > 2$,*

$$\frac{\delta \log \varphi(\Delta_1)}{\Delta_1 \delta \Delta_1} = \frac{\delta \log \varphi(\Delta_2)}{\Delta_2 \delta \Delta_2} = \quad \cdot \cdot = k \text{ suppose}$$

Clearing of fractions and integrating,

$$\varphi(\Delta) = ce^{\frac{1}{2} k \Delta^2}$$

* When $n = 2$ it reduces to an identity

where e is the base of the Napierian system of logarithms and c is a constant .

Now, since ψ is to be a maximum, $\dfrac{d^2\psi}{d\,\Gamma^2}$ must be negative But when ψ is a maximum subject to the condition $[\varDelta]=0$, then $\dfrac{d^2\psi}{d\,\Gamma^2}=nk\psi$ Hence, since ψ is positive, k must be negative, and, putting it equal to $-\dfrac{\mathrm{I}}{\mu^2}$, we have

$$\varphi(\varDelta)=ce^{-\frac{\varDelta^2}{2\mu^2}}$$

the law of error sought

16 In this expression there are two symbols undetermined, c and μ To find c Since it is certain that all of the errors lie between the maximum errors $+a$ and $-a$, we have

$$c\int_{-a}^{+a}e^{-\frac{\varDelta^2}{2\mu^2}}d\varDelta=\mathrm{I}$$

But as the values of a are different for different kinds of observations, and as we cannot in general assign these values definitely, we must take $+\infty$ and $-\infty$ as the extreme limits of error, so that c is found from

$$c\int_{-\infty}^{+\infty}e^{-\frac{\varDelta^2}{2\mu^2}}d\varDelta=\mathrm{I}$$

and hence (see Art 6)

$$c=\frac{\mathrm{I}}{\mu\sqrt{2\pi}}$$

and the law of error may be written

$$\varphi(\varDelta)=\frac{\mathrm{I}}{\mu\sqrt{2\pi}}\,e^{-\frac{\varDelta^2}{2\mu^2}}$$

or by putting $h^2=\dfrac{\mathrm{I}}{2\mu^2}$

$$\varphi(\varDelta)=\frac{h}{\sqrt{\pi}}\,e^{-h^2\varDelta^2}$$

When this latter form is used it is only for greater conveni-
ence in writing, and h is to be looked on as a mere sym-
bol standing for $\dfrac{1}{2\mu^2}$

As regards μ^2, it is evident that for $e^{-\frac{\Delta^2}{2\mu^2}}$ to be a possible

quantity $\dfrac{\Delta^2}{\mu^2}$ must be an abstract number   Hence μ is a

quantity expressed in the same unit of measure as Δ.

Also, from the form of the function $\varphi(\Delta)$, it is evident
that the probability of an error Δ will be the larger the
larger μ is, and *vice versa*   Hence μ is a test of the
quality of observations of different series, the unit being
the same.

Again, the total number of errors in a series being n, the
number between Δ and $\Delta + d\Delta$ will, from the definition of
probability, be $n\,\varphi(\Delta)\,d\Delta$ Hence the sum of the squares of
the errors Δ in the same interval will be equal to $n\Delta^2\,\varphi(\Delta)\,d\Delta$,
and the sum of the squares of the errors between the limits
of error $+a$ and $-a$ will be

$$n \int_{-a}^{+a} \Delta^2\,\varphi(\Delta)\,d\Delta$$

Extending the limits of error $\pm a$ to $\pm \infty$, this expression
becomes, after substituting for $\varphi(\Delta)$ its value,

$$\frac{n}{\mu\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Delta^2\, e^{-\frac{\Delta^2}{2\mu^2}}\, d\Delta$$

which (see Art 6) reduces to $n\mu^2$

Hence μ^2 is the mean of the sum of the squares of the
errors Δ that occur in the series   It is called the *mean-
square error*, and will be referred to as the m. s e.

17. **The Principle of Least-Squares.**—Having de-
fined the symbols c and μ in the expression for $\varphi(\Delta)$, let us
return to Art 15, Eq 5

If the observed values are of the same quality through-
out, μ is constant and the product ψ becomes $c^n e^{-\frac{[\Delta^2]}{2\mu^2}}$ This

product is evidently a maximum when $[\varDelta^2]$ is a minimum, that is, *if we assume that each of a very large number of observed values of a quantity is of the same quality, the most probable value of the quantity is found by making the sum of the squares of the errors a minimum*

If the observed values are not of the same quality, μ is different for the different observations, and the most probable value of the unknown would be found from the maximum value of $e^{-\left[\frac{\varDelta^2}{2\mu^2}\right]}$, that is, from the minimum value of $\left[\frac{\varDelta^2}{\mu^2}\right]$ Thus *if each of a large number of observed values of a quantity is of different quality, the most probable value of the quantity is found by dividing each error of observation by its m s e and making the sum of the squares of the quotients a minimum*

This latter includes the case in which the observed quantity is a function of several independent unknowns whose values are to be found For if

$$M_1 = f_1(x, y, \qquad)$$
$$M_2 = f_2(x, y, \qquad)$$
.

where the functions $f_1, f_2, \ldots$ are of the same form and differ only in the constants that enter, then if $\varDelta_1, \varDelta_u, \ldots$ denote the errors of observation, we have

$$\varDelta_1 = f^1(x, y, \quad . \quad) - M_1$$
$$\varDelta_2 = f^2(x, y, \ldots) - M_2$$
. . . .

Hence since $\varDelta_1, \varDelta_2,$ are functions of the independent variables $x, y,$. we must have

$$\left[\frac{1}{\mu^2}\{f(x, y, \qquad) - M\}^2\right] = \text{a minimum}$$

with respect to the variables $x, y,$. But the differentiation of this equation with respect to $x, y,$. and the equating of the differential coefficients to zero, gives as

7

many equations as unknowns, from which equations the most probable values of $x, y,$ may be found

18. Two other inferences from the preceding general principles are important

(a) Since in a series of observed values of different quality the sum of the squares of the errors, divided by their respective m s e made a minimum, leads to the most probable values, it follows that observed values of different quality are put on a common basis for combination by dividing by their respective m s e   This conclusion will be found developed in Chapter III

(b) Since $\dfrac{\Delta^2}{\mu^2}$ is an abstract number, no matter what the unit of measure in which the observed values are expressed, it follows that *heterogeneous measures* may be combined in the same minimum equation. For an example in which this is fully brought out see Art 169 ﹗

The Law of Error of a Linear Function of Independently Observed Quantities.

19 We have found the law of error in the case of a quantity directly observed, and which may be a function of one or more unknowns. There remains the question as to the form the law of error assumes in the case of a quantity, F, which is a linear function of several independently observed quantities, $M_1, M_2, \ldots M_n$, that is, when

$$F = a_1 M_1 + a_2 M_2 + \quad . . + a_n M_n$$

where $a_1, a_2, .$. are constants.

For simplicity in writing consider two observed quantities, $M_1, M_2,$ only, and let μ_1, μ_2 be their m s e. The probability of the simultaneous occurrence of the errors Δ_1 in M_1 and Δ_2 in M_2 is

$$\frac{h_1 h_2}{\pi} e^{-h_1^2 \Delta_1^2 - h_2^2 \Delta_2^2} d\Delta_1 \, d\Delta_2 \qquad (1)$$

Now, an error Δ_1 in M_1 and an error Δ_2 in M_2 produce an error Δ in F, accoiding to the relation

$$\Delta = a_1\Delta_1 + a_2\Delta_2 \tag{2}$$

and this relation can always be satisfied by combining any value of Δ_2 with all values of Δ_1 ranging from $-\infty$ to $+\infty$ The probability, theiefore, of an error Δ in F may be written

$$\varphi(\Delta)\, d\Delta = \frac{h_1 h_2}{\pi}\, d\Delta_2 \int_{-\infty}^{+\infty} e^{-h_1{}^2\Delta_1{}^2 - h_2{}^2\Delta_2{}^2}\, d\Delta_1$$

But from (2), and since Δ_2 is independent of Δ_1,

$$d\Delta = a_2\, d\Delta_2$$

Hence

$$\varphi(\Delta)\, d\Delta$$

$$= \frac{h_1 h_2}{\pi}\frac{d\Delta}{a_2} \int_{-\infty}^{+\infty} e^{-h_1{}^2\Delta_1{}^2 - h_2{}^0\left(\frac{\Delta - a_1\Delta_1}{a_2}\right)^2}\, d\Delta_1$$

$$= \frac{h_1 h_2}{\pi}\frac{d\Delta}{a_2} e^{-\frac{h_1{}^2 h_2{}^2}{h_1{}^2 a_2{}^2 + h_2{}^2 a_1{}^2}\Delta^2} \int_{-\infty}^{+\infty} e^{-\frac{h_1{}^2 a_2{}^2 + h_2{}^2 a_1{}^2}{a_2{}^2}\left(\Delta_1 - \frac{h_1{}^2 a_1}{h_1{}^2 a_2{}^2 + h_2{}^2 a_1{}^2} \Delta\right)^2}\, d\Delta_1$$

$$= \frac{h_1 h_2}{\sqrt{(h_1{}^2 a_2{}^2 + h_2{}^2 a_1{}^2)}\,\pi} e^{-\frac{h_1{}^2 h_2{}^2}{h_1{}^2 a_2{}^2 + h_2{}^2 a_1{}^2}\Delta^2}\, d\Delta$$

which is of the form

$$\frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}\, d\Delta$$

That is, *the law of error of the function F is the same as that of the independently measured quantities M_1, M_2*

The m s e of the function F is found fiom

$$h^2 = \frac{h_1{}^2 h_2{}^2}{h_1{}^2 a_2{}^2 + h_2{}^2 a_1{}^2}$$

that is, from

$$\mu^2 = a_1{}^2\mu_1{}^2 + a_2{}^2\mu_2{}^2$$
$$= [a^2\mu^2]$$

This theorem is one of the most important in the method of least squares, and will be often iefeiied to.

Ex To find the m s e of the arithmetic mean of n equally well observed values of a quantity

We have

$$F = \frac{1}{n}(M_1 + M_2 + \quad\quad + M_n)$$

Let

μ_o = m s e of the arithmetic mean
μ = m s e of each observed value

Then

$$\mu_o^2 = \frac{1}{n^2}(\mu^\circ + \mu^2 + \quad . \quad \text{to } n \text{ terms})$$

or

$$\mu_o = \frac{\mu}{\sqrt{n}}$$

That is, *the m s e of the arithmetic mean of n observations is* $\dfrac{1}{\sqrt{n}}$ *part of that of a single observation*

On the Comparison of the Accuracy of Different Series of Observations

20 The Mean-Square Error.—We have seen in Art. 16 that the m s e μ affords a test of the relative accuracy of different series of observations   This test was suggested by the fundamental formula of the law of error, and is naturally the first that would be taken for that purpose

The value of μ^2 is the mean of the sum of the squares of the errors in a series between the extreme limits of error, and since the probability of an error is the number of cases favorable to its occurrence divided by the total number of cases, μ^2 is given by the expression

$$\int_{-a}^{+a} \Delta^2 \, \varphi(\Delta) \, d\Delta$$

where $+a$ and $-a$ are the limits of error.

Hence if the number of errors n is a very large number a close approximation to the value of μ^2 will be given by

$$\mu^2 = \frac{\Delta_1^2}{n} + \frac{\Delta_2^2}{n} + \quad\quad + \frac{\Delta_n^2}{n}$$
$$= \frac{[\Delta^2]}{n}$$

The difference in precision of these two values of μ^2 will be pointed out later   (See Art 23)

21 The Probable Error.—A second method of determining the relative precision of different series of observations is by comparing errors which occupy the same relative position in the different series when the errors are arranged in order of magnitude   The errors which occupy the middle places in each series are, for greater convenience, the ones chosen

Let the errors in a series, arranged in order of magnitude, be

$$\pm 2a, \qquad \pm r, \quad . \quad o,$$

each error being written as many times as it occurs, then we give to that error r which occupies the middle place, and which has as many errors numerically greater than it as it has errors less than it, the name of *probable error*. If, therefore, n is the total number of errors, the number lying between $+r$ and $-r$ is $\frac{n}{2}$, and the number outside these limits is also $\frac{n}{2}$. In other words, the probability that the error of a single observation in any system will fall between the limits $+r$ and $-r$ is $\frac{1}{2}$, and the probability that it will fall outside these limits is also $\frac{1}{2}$ We have, therefore,

$$\frac{h}{\sqrt{\pi}} \int_{-r}^{+r} e^{-h^2 \Delta^2} d\Delta = \frac{1}{2}$$

from which to find r

If we put $h\Delta = t$, and the value $t = \rho$ corresponds to $\Delta = r$, then

$$\frac{2}{\sqrt{\pi}} \int_{0}^{\rho} e^{-t^2} dt = \frac{1}{2}$$

Expanding the integral in a series as in (c) Art 6, we shall find that approximately the resulting equation is satisfied by

$$\rho = 0\ 47694$$

Now, since

$$hr = \rho = 0\ 47694 \text{ and } h\mu \sqrt{2} = 1$$

it follows that

$$r = 0\ 6745\mu$$
$$= \frac{2}{3}\mu \text{ roughly}$$

Hence to find the probable error we compute first the mean-square error and multiply it by 0 6745

As a check, the error which occupies the middle place in the series of errors arranged in order of magnitude may be found It will be nearly equal to the computed value, if the series is of moderate length

It is to be clearly understood that the term probable error does not mean that that error is more probable than any other, but only that in a future observation the probability of committing an error greater than the probable error is equal to the probability of committing an error less than the probable error Indeed, of any single error the most probable is zero Thus the probability of the error zero is to that of the probable error r as

$$\frac{h}{\sqrt{\pi}} \quad \frac{h}{\sqrt{\pi}} e^{-h^2 r^2}$$

or

$$1 \quad e^{-(0.47694)^2}$$

or

$$1 : 08$$

The idea of probable error is due to Bessel (*Berlin Astron Jahrb* , 1818) The name is not a good one, on account of the word probable being used in a sense altogether different from its ordinary signification It would be better to use the term *critical error*, for example, as suggested by De Morgan, or *median error*, as proposed by Cournot

22 **The Average Error.**—It naturally occurs, as a third test of the accuracy of different series of observations, to take the mean of all the positive errors and the mean of all the negative errors, and then, since in a large number of

observations there will be nearly the same number of each kind, to take the mean of the two results without regard to sign This gives what may be termed the *average error* It is usually denoted by the Greek letter η

Reasoning as in Art 20, we have approximately

$$\eta = \frac{[\varDelta}{n}$$

where $[\varDelta$ is the arithmetic sum of the errors

An expression for η in terms of the mean-square error μ may be found as follows The number of errors between $\varDelta$ and $\varDelta + d\varDelta$ is

$$n \, \varphi(\varDelta) \, d\varDelta$$

and the sum of the positive errors in the series is

$$n \int_0^\infty \varDelta \, \varphi(\varDelta) \, d\varDelta$$

The sum of the negative errors being the same, the sum of all the errors is

$$2n \int_0^{+\infty} \varDelta \, \varphi(\varDelta) \, d\varDelta$$

Hence

$$\eta = 2 \int_0^{+\infty} \varDelta \, \varphi(\varDelta) \, d\varDelta$$

$$= \frac{2h}{\sqrt{\pi}} \int_0^{+\infty} \varDelta e^{-h^2 \varDelta^2} \, d\varDelta$$

$$= \frac{1}{h \sqrt{\pi}} = \mu \sqrt{\frac{2}{\pi}}$$

the relation required

The average error may, as stated above, be directly used as a test of the relative accuracy of different series of observations Indeed, I think it should be more used for this purpose than it is The general custom is, however, to employ it as a stepping-stone to find the mean-square and probable errors This can be done, for the reason that it is more easy to compute $[\varDelta$ than $[\varDelta^2]$

The formulas for μ and r computed in this way are as follows   From the last equation preceding

$$\mu = \sqrt{\frac{r}{2}}\,\eta$$

$$= 1\,2533\,\frac{[\varDelta}{n}$$

and from Art 21

$$r = 0\,6745\,\mu$$

$$= 0\,8453\,\frac{[\varDelta}{n}$$

The relations connecting μ, r, and η are easily remembered in the following form

$$\mu\,\sqrt{2} = \frac{r}{\rho} = \sqrt{\pi}\,\eta$$

These relations may also be conveniently arranged in tabular form

	μ	r	η
$\mu =$	1.0000	1 4826	1 2533
$r =$	0 6745	1.0000	0 8453
$\eta =$	0 7979	1.1829	1 0000

23. We have seen that the m s. e μ may be computed from the sum of the squares of the errors and also from the sum of the errors without regard to sign   In the derivation of each formula certain approximations have been made   The question then arises which of the two methods will give the more reliable result   This will be shown by the ranges in the two determinations of the value of μ

(a) We proceed to find first the m s e in the determination of μ^2 by taking the approximate formula

$$\frac{[\varDelta^2]}{n}\,(= \mu_1^2\ \text{suppose})$$

instead of the rigid formula

$$\int_{-\infty}^{+\infty} \varDelta^2\, \varphi(\varDelta)\, d\varDelta$$

If we put

$$\lambda = \mu_1^2 - \mu^2$$

then, since μ^2 is the exact value, λ will be the error of μ^2 computed from the errors $\varDelta_1, \varDelta_2$. according to the formula $\dfrac{[\varDelta^2]}{n}$ Squaring, we have

$$\lambda^2 = \mu_1^4 - 2\mu_1^2\mu^2 + \mu^4$$
$$= \frac{[\varDelta^4]}{n^2} + \frac{2}{n^2}(\varDelta_1^2\varDelta_2^2 + \varDelta_1^2\varDelta_3^2 + \ldots) - 2\mu^2\frac{[\varDelta^2]}{n} + \mu^4$$

Now, letting the errors $\varDelta$ assume all possible values, the average value of the fourth powers is (see Art 6)

$$\frac{2h}{\sqrt{\pi}}\int_0^\infty \varDelta^4 e^{-h^2\varDelta^2} d\varDelta = \frac{3}{4h^4}$$
$$= 3\mu^4$$

The number of the products $\varDelta_1^2\varDelta_2^2$, $\varDelta_1^2\varDelta_3^2$, being the number of combinations of n things, two at a time, is $\dfrac{n(n-1)}{2}$ and the average of the values is

$$\frac{n(n-1)}{2}\left\{ \frac{2h}{\sqrt{\pi}}\int_0^\infty \varDelta^2 e^{-h^2\varDelta^2} d\varDelta \right\}^2$$

that is (Art 20),

$$\frac{n(n-1)}{2}\mu^4$$

The average value of $\mu^2\dfrac{[\varDelta^2]}{n}$ is μ^4 Hence finally

$$\lambda^2 = \frac{3\mu^4}{n} + \frac{2}{n^2}\frac{n(n-1)}{2}\mu^4 - 2\mu^4 + \mu^4$$
$$= \frac{2\mu^4}{n}$$

8

and

$$\mu_1^2 = \mu^2 \pm \mu^2 \sqrt{\frac{2}{n}}$$

or

$$\mu_1 = \mu \left(1 \pm \sqrt{\frac{2}{n}} \right)^{\frac{1}{2}}$$

$$= \mu \left(1 \pm \frac{1}{\sqrt{2n}} \right) \text{ when } n \text{ is very large}$$

(b) In the second place, for the average error η we proceed in precisely the same way   We have

$$\eta = 2 \int_0^\infty \Delta \varphi(\Delta) \, d\Delta$$

$$\eta_1 = \frac{[\Delta]}{n}$$

Let $\lambda_1 = \eta_1 - \eta$

Then $\lambda_1^2 = \eta_1^2 - 2\eta_1 \eta + \eta^2$

$$= \frac{[\Delta^2]}{n^2} + \frac{2}{n^2}(\Delta_1 \Delta_2 + \Delta_1 \Delta_3 + \qquad) - 2\eta \frac{[\Delta]}{n} + \eta^2$$

$$= \frac{\mu^2}{n} + \frac{2}{n^2} \frac{n(n-1)}{2} \frac{2\mu^2}{\pi} - \eta^2$$

$$= \frac{\mu^2}{n} \left(1 - \frac{2}{\pi} \right)$$

which gives the error in η
  Also, since

$$\mu = \sqrt{\frac{\pi}{2}} \, \eta$$

the error in μ is

$$\frac{\mu}{\sqrt{n}} \sqrt{\frac{\pi}{2}} \sqrt{1 - \frac{2}{\pi}}, \text{ that is, } \mu \sqrt{\frac{\pi - 2}{2n}}$$

Hence by this method of computing μ the value of μ is contained between the mean limits

$$\mu \left(1 \pm \sqrt{\frac{\pi - 2}{2n}} \right)$$

Now, since $\pi - 2 > 1$, the limits in the latter case are the larger, and we therefore conclude that the former method of computing μ is the better of the two

24 From the equation

$$\mu = \eta \sqrt{\frac{\pi}{2}}$$

we may derive a test of the validity of the law of error, and a rather curious one   For μ and η may be determined from measurements, and if the experimental values found satisfy the equation

$$\frac{\mu^2}{\eta^2} = \frac{\pi}{2}$$

we must conclude that the theory is correct. This may be classed as an additional *à posteriori* proof to that given in Art 51

25 Whether we should use the m s e or the p e in stating the precision is largely a matter of taste   Gauss says " The so-called probable error, since it depends on hypothesis, I, for my part, would like to see altogether banished , it may, however, be computed from the mean by multiplying by o 6744897 " On the other hand, the International Committee of Weights and Measures decided in favor of the probable error " It has been thought best in this work that the measure of precision of the values obtained should always be referred to the probable error computed from Gauss' formula, and not to the mean error " (*Procès Verbaux*, 1879, p 77)

In the United States, in the Naval Observatory, the Coast Survey, the Engineer Corps, and the principal observatories, the p e is used altogether   So, too, in Great Britain, in the Greenwich Observatory, the Ordnance Survey, etc   In the G T. Survey of India the m s e is used, for the reason given by Gauss above   Among German geodeticians and astronomers the m s e is very generally employed.

In this book the m s e will be used in the text, and the m s, e and p. e in the examples indifferently.
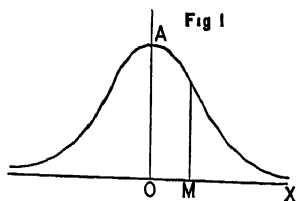
The Probability Curve

26 The principles laid down in the preceding articles may be illustrated geometrically as follows

We have seen that in a series of observations the probability that an error will lie between the values x and $x + dx$ is given by the expression

$$\frac{h}{\sqrt{\pi}} \, e^{-h^2 x^2} \, dx$$

Now, if O is the origin of co-ordinates, and a series of errors, x, are represented by the distances from O along the axis of abscissas OX, positive errors being taken to the right of O and negative errors to the left, then the probability, in a future observation, of an error falling between x and $x + dx$ will be represented by the rectangle whose height is $\frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$ and width dx, or, more strictly, by the ratio of this rectangle to the sum of all such rectangles between the extreme limits of error   This sum we have for convenience already denoted by unity

Hence for a series of observations whose quality is known, by giving to x all values from $+\infty$ to $-\infty$ and drawing the corresponding ordinates, we shall have a continuous curve whose equation may be written

$$y = \frac{h}{\sqrt{\pi}} \, e^{-h^2 x^2}$$

This curve is called the probability curve

27 **To Trace the Form of the Curve.**—Since x enters to the second power, and y to the first power, the curve is symmetrical with respect to the axis of y, and the form of the equation shows that it lies altogether on one

side of the axis of x Also, when $x = 0$, $\dfrac{dy}{dx} = 0$, that is, the tangent at the vertex is parallel to the axis of x.

As x increases from o the values of y continually decrease   When $x = \pm \infty$, then

$$y = 0 \text{ and } \frac{dy}{dx} = 0$$

showing that the axis of x is an asymptote
    Again, since

$$\frac{d^2y}{dx^2} = \frac{2h^3}{\sqrt{\pi}} \, e^{-h^2x^2}(2h^2x^2 - 1)$$

there is a point of inflection when

$$x = \frac{1}{h\sqrt{2}} = \mu$$

and the m s e is therefore the abscissa of the point of inflection   Also, when $x = 0$, $\dfrac{d^2y}{dx^2}$ is negative, showing that the ordinate at the vertex is the maximum ordinate   Hence the curve is of the form indicated in Fig 1, OA representing the maximum ordinate and OM the m s. e.

    The values of h, that is, of $\dfrac{1}{\mu\sqrt{2}}$, being different for different series of observations, the form of the curve will change for each series, and the curve may be plotted to scale from values of y corresponding to assumed values of x

    In plotting the curve, since the maximum ordinate at the vertex $\dfrac{h}{\sqrt{\pi}}$ enters as a factor into the values of each of the other ordinates, its value may be arbitrarily assumed We may therefore adopt a scale for plotting the ordinates different from the scale by which the abscissas are plotted, in order to show the curve more clearly
    The form of the curve is in accordance with the principles already laid down in deducing the law of error, and

could have been derived from them directly   Thus that small errors are more probable than large is indicated by the element rectangle areas being greater for values of x near zero than for values more distant, that very large errors have a very small probability is indicated by the asymptotic form of the curve, and that positive and negative errors are equally probable is indicated by its symmetrical form with respect to the axis of y

28 The area of the curve of probability being the sum of the rectangles $\dfrac{h}{\sqrt{\pi}} e^{-h^2 x^2} dx$, for values of x extending from $+\infty$ to $-\infty$ we have denoted by unity   If, then, we represent by the area of this curve the total number of errors that occur in a series of observations, it follows from the definition of probability that the area included between certain assigned limits would represent the number of errors to be expected in the series between the values of those limits.

Fig 2
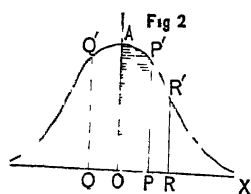
Thus if O is the origin, the area to the right of OA would represent the number of positive errors and the area to the left of OA the number of negative errors   The area $OPP'A$ would represent the number of positive errors less than OP, the area $PRR'P'$ the number that lie between OP and OR

If the area $AOPP'$ is equal to one-half the total area AOX then the number of positive errors less than OP would be equal to the number greater than OP Hence OP would represent the probable error   If OQ be taken equal to OP, the area $PQQ'P'$ would represent the number of errors numerically less than the probable error

29 The average error η may be illustrated as follows

If x_1 is the abscissa of the centre of gravity of any curve referred to the axes of X and Y, then

$$x_1 = \frac{\int yx\, dx}{\int y\, dx}$$

Applying this to the curve of probability and calling the whole area unity, we have

$$x_1 = \int_{-\infty}^{+\infty} \frac{h}{\sqrt{\pi}} e^{-h^2 x^2} x \, dx$$

$$= \frac{1}{h\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} t \, dt \text{ if } hx = t$$

$$= \frac{1}{h\sqrt{\pi}}$$

$$= \eta, \text{ the average error}$$

The Law of Error applied to an Actual Series of Observations.

We here bridge over the gulf between the ideal series from which we have derived the law of error and the actual series with which we have to deal in practical work, and which can only be expected to come partially within the range of the law constructed for the ideal

30 **Effect of Extending the Limits of Error to** $\pm \infty$.—The expression $\frac{h}{\sqrt{\pi}} e^{-h^2 \Delta^2} d\Delta$ gives the value of the probability of an error between Δ and $\Delta + d\Delta$ in an ideal series of observations where the values are continuous between limits infinitely great   In all actual series the possible error is included within certain finite limits, and the probability of the occurrence of an error beyond those limits is zero   Practically, however, the extension of the limits of error to $\pm \infty$ can make no appreciable difference in either case, as the function $\varphi(\Delta)$ decreases so rapidly that we can regard it as infinitesimal for large values of Δ; in other words, the greater number of errors is in the neighborhood of zero, and therefore the most important part is the part covered by both   This has been illustrated geometrically in the discussion of the probability curve, and will now be developed from another point of view

We have for the probability of the occurrence of an error not greater than a, in a series of observations,

$$\frac{2h}{\sqrt{\pi}} \int_0^a e^{-h^2\Delta^2}\, d\Delta$$

This may be put in the form $(t = h\Delta)$

$$\frac{2}{\sqrt{\pi}} \int_0^t e^{-\mu}\, dt$$

and is usually denoted by the symbol $\theta(t)$.

The method of evaluating $\theta(t)$ has been explained in (c) Art 6   In Table I will be found the values of the function $\theta(t)$ corresponding to the argument $\dfrac{ah}{\rho}$, that is, $\dfrac{a}{r}$ The reason for arranging the table in this way is that it is more convenient to compute $\dfrac{a}{r}$ than $\dfrac{a}{r}\rho$

The probability that an error exceeds a certain error a is $1 - \theta(t)$, and may be found from Table I   by deducting the tabular value from unity   Thus we have the probability that a is greater than r is 0 5, than $2r$ is 0.1773, than $3r$ is 0 0430, than $4r$ is 0.0070, than $5r$ is 0 0007, than $6r$ is 0.0001.

Hence in 10,000 observations we should expect only one error greater than $6r$, in 1,000 only one greater than $5r$, in 100 only one greater than $4r$, and in 25 only one greater than $3r$ If in any set of observations we found results much at variance with these we could assume that they arose from some unusual cause, and should, therefore, be specially examined   As in practice the number of observations in any case is usually under 100, we are eminently safe in taking the maximum error at about $5r$ or 3μ.

We are now in a position to estimate the change introduced by replacing c as found from

$$c \int_{-a}^{+a} e^{-h^2\Delta^2}\, d\Delta = 1$$

by $\dfrac{h}{\sqrt{\pi}}$ as found from

$$\iota \int_{-\infty}^{'+\infty} \iota^{-h\Delta^2}\, d\Delta = 1$$

in the investigation of the law of error

These equations may be written

$$2c \int_{0}^{a} e^{-h^2\Delta^2}\, d\Delta = 1$$

$$\frac{2h}{\sqrt{\pi}} \left(\int_{a}^{\infty} \iota^{-h^2\Delta^2}\, d\Delta + \int_{0}^{a} e^{-h^2\Delta^2}\, d\Delta \right) = 1$$

Hence

$$c = \frac{h}{\sqrt{\pi}} \left(1 + \frac{2h}{\sqrt{\pi}} \int_{a}^{\infty} \iota^{-h^2\Delta^2}\, d\Delta \right) \text{ approximately}$$

and taking $a = 5r$, we have, from Table I ,

$$\iota = \frac{h}{\sqrt{\pi}} (2 - 0\ 9993)$$

$$= 1\ 001 \frac{h}{\sqrt{\pi}}$$

Hence the difference being less than $\dfrac{1}{1000}$ of the quantity sought, the approximate value of c found by extending the limits $\pm a$ to $\pm \infty$ may be considered satisfactory

31 **Various Laws of Error.**—We have taken the arithmetic mean of a series of observed values of a quantity made under like conditions as the most plausible value of the quantity   The supposition of each observed quantity being subject to the same law of error leads to the mean as the most probable value   " The method of least squares is, in fact, a method of means, but with some peculiar characters   The method proceeds upon this supposition, that all errors are not equally probable, but that small errors are more probable than large ones " *

Now, in an ordinary series we assume a good deal when we take each observation of the series as subject to the same

* Whewell, *History of the Inductive Sciences*, vol 11

special law of error—the exponential law   We can certainly conceive of laws different from this one   It is more probable that each set of observations has its own law depending on instrument, observer, and conditions   If we could go back to the sources of error we could find this law in each case. Let us follow out this idea in a few simple cases and see to what it leads ·

32 First take the case where all errors are equally probable—that is, where $\varDelta$ can with the same probability assume all values between $+a$ and $-a$, the extreme limits of error

Since a is the maximum error,

$$\varphi(\varDelta)\int_{-a}^{+a}d\varDelta = 1,\ \varphi(\varDelta)\ \text{being constant,}$$

and therefore

$$\varphi(\varDelta) = \frac{1}{2a}$$

the law of error.

For the m s. e μ we have

$$\mu^2 = \int_{-a}^{+a}\frac{\varDelta^2}{2a}\ d\varDelta$$

$$= \frac{a^2}{3}$$

Also

$$\eta = \int_{-a}^{+a}\varphi(\varDelta)\ \varDelta\ d\varDelta$$

$$= \int_{-a}^{+a}\frac{\varDelta\ d\varDelta}{2a}$$

$$= \frac{a}{2}$$

The p e is found from the relation

$$\int_{-r}^{+r}\varphi(\varDelta)\ d\varDelta = \frac{1}{2}$$

or

$$\int_{-r}^{+r}\frac{d\varDelta}{2a} = \frac{1}{2}$$

and $r = \dfrac{a}{2}$

that is, the p e is half the max. error

This is also evident geometrically, for the curve of error

$$y = \text{constant}$$

is a straight line parallel to the axis of abscissas

Hence by definition we find the p e , by bisecting the area, to be $\dfrac{a}{2}$ The p e would be represented in the figure by OP

Fig 3
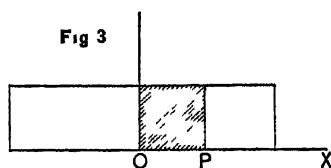
33 Next consider an error to arise from two independent sources, x, y, each of which can with the same probability assume all values between $+a$ and $-a$, so that for the total error $\varDelta$ we have

$$\varDelta = x + y$$

To find $\varphi(\varDelta)$, the law of error If $\varDelta$ and $\varDelta + d\varDelta$ denote two consecutive errors, then since x and y have values between $+a$ and $-a$ with interval $d\varDelta$, each of the quantities x, y has $\dfrac{2a}{\varDelta}$ possible values, and the whole number of possible causes

of error is $\dfrac{2a}{d\varDelta} \times \dfrac{2a}{d\varDelta}$ or $\dfrac{4a^{2}}{(d\varDelta)^{2}}$. The causes which are favorable for an error of the magnitude $+\varDelta$ answer to all possible values of x, y which together give $+\varDelta$, namely,

for x, $\varDelta - a$, $\varDelta - a + d\varDelta$, $\varDelta - a + 2d\varDelta$, $a - d\varDelta$, a

for y, a, $a - d\varDelta$, $a - 2d\varDelta$,   . $\varDelta - a + d\varDelta$, $\varDelta - a$

and whose number is therefore $\dfrac{2a - \varDelta}{d\varDelta}$

In the same way we find the number of causes favorable for the error of the magnitude $-\varDelta$ to be $\dfrac{2a + \varDelta}{d\varDelta}$

Hence the probability $\varphi(\varDelta)\,d\varDelta$ of an error between $\varDelta$ and $\varDelta + d\varDelta$ is given by

$$\varphi(\varDelta)\,d\varDelta = \frac{2a \pm \varDelta}{d\varDelta} - \frac{4a^2}{(d\varDelta)^2}$$

$$= \frac{2a \pm \varDelta}{4a^2}d\varDelta$$

that is,

$$\varphi(\varDelta) = \frac{2a + \varDelta}{4a^2} \text{ when } \varDelta \text{ lies between } -2a \text{ and } 0$$

$$\varphi(\varDelta) = \frac{2a - \varDelta}{4a^2} \text{ when } \varDelta \text{ lies between } 0 \text{ and } +2a.$$

For the m s e. we have

$$\mu^2 = \int_0^{2a} \varDelta^2 \frac{2a - \varDelta}{4a^2}\,d\varDelta + \int_{-2a}^0 \varDelta^2 \frac{2a + \varDelta}{4a^2}\,d\varDelta$$

$$= \frac{2}{3}a^2$$

For the p. e ,

$$\int_0^r \frac{2a - \varDelta}{4a^2}d\varDelta + \int_{-r}^0 \frac{2a + \varDelta}{4a^2}\,d\varDelta = \frac{1}{2}$$

$$r = (2 - \sqrt{2})\,a$$

Geometrically, the equations

$$y = 2a - x$$
$$y = 2a + x$$

Fig 4
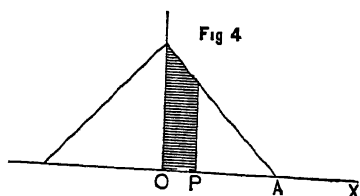
represent two straight lines which cut the axes of x and y at an angle of $45°$, and therefore the curve of probability is as in the figure If in the figure $OA = 2a$, then OP represents the p e

34. From the preceding we may derive an important practical point In an ordinary seven-place log. table the seventh place is never in error by more than 0 5 Hence,

this being the maximum error, the p e of a log as given in the tables is o 25 in the seventh place The interpolated value at the greatest distance from a tabular value is the mean between two tabular values Its p e , from Art 33, is

$$(2 - \sqrt{2}) \times 0\,25 = 0\,15$$

Hence the p e. of the log of a number may be taken o 2 in the seventh decimal place The p e of the number corresponding to this log is (Ex 1, Art 7)

$$\frac{2}{10^8 \times \text{mod}} = \frac{1}{22 \times 10^6} \text{ approx}$$

Suppose now that we are computing a chain of triangles starting from a measured base The p c. of the base may be taken as $\frac{1}{1,000,000}$ of its length. Hence the error arising from this source is 22 times that to be expected from the log. tables Again, the triangulation will be most exact, and therefore the test most severe, when the angles of each triangle are equal to 60° Now, the change in log sin 60" corresponding to a change of 1″ is 12 2 in units of the seventh decimal place And in a primary triangulation an angle may, with the instruments now in use, be measured with a p e of o″ 25 Hence

p e of log sin 60 = 12 2 × 0 25

= 3 0 in units of the seventh decimal place,

which p c is 15 times greater than that arising from the log tables

For the solution of triangles, therefore, we conclude that, with our present means of measurement, seven-place tables are sufficient The common practice is to carry out to eight places to give greater accuracy in the seventh place, and then drop the eighth place in stating the final result (See Struve, *Arc du Méridien*, vol 1 p 94)

35 If an error Δ arises from three independent sources of the same kind, each of which can with the same probability assume all values between $+a$ and $-a$, then, the

maximum error being $3a$, we have, from similar reasoning to that employed in Art. 33,

$$\varphi(\varDelta) = \frac{(3a - \varDelta)^3}{16a^3} \text{ when } \varDelta \text{ is between } + 3a \text{ and } + a$$

$$\varphi(\varDelta) = \frac{3a^2 - \varDelta^2}{8a^3} \text{ when } \varDelta \text{ is between } + a \text{ and } - a$$

$$\varphi(\varDelta) = \frac{(3a + \varDelta)^3}{16a^3} \text{ when } \varDelta \text{ is between } - a \text{ and } - 3a$$

Also

$$\mu^2 = 2 \int_0^a \frac{3a^2 - \varDelta^2}{8a^3} \varDelta^2 d\varDelta + 2 \int_a^{3a} \frac{(3a - \varDelta)^3}{16a^3} \varDelta^2 d\varDelta$$

$$= a^2$$

and

$$\int_{-r}^{+r} \frac{3a^2 - \varDelta^2}{8a^3} d\varDelta = \frac{1}{2}$$

$$r = 0.71a$$

The curve of probability consists of three parts, as in the figure:

Fig 5
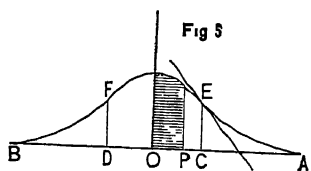
$OA = OB = 3a \qquad OC = OD = a$
There are common tangents to the two branches at E and F, and the curve touches the axis of X at A and B The p e is represented by OP

36 A consideration of the results obtained in Arts 31–35 will show that the more numerous the sources of error assumed the nearer we approach the results obtained from the Gaussian law of error. Thus for

one source $\dfrac{r}{\mu} = 0.87$ $\dfrac{r}{\eta} = 1.00$

two sources $\dfrac{r}{\mu} = 0.72$ $\dfrac{r}{\eta} = 0.88$

three sources $\dfrac{r}{\mu} = 0.71$ $\dfrac{r}{\eta} = 0.87$

Gaussian law $\dfrac{r}{\mu} = 0.67$ $\dfrac{r}{\eta} = 0.85$

The forms of the curves of probability show the same approach to coincidence Starting with a straight line as the cui ve for a single source of error, we appioach quite closely to the Gaussian probability curve, even with so small a number of souices of eiror as thiee Hence we should expect that, starting from the postulate of an error being derived fiom the combined influence of a very large number of independent sources of erior, we should arrive at the Gaussian law of error A complete demonstiation of this by Bessel, to whom the idea itself is due, will be found in *Astron Nachr*, Nos 358, 359 The elementary proof given in Ait 33 for the simple case of two souices of error is due to Zachariæ

37 **Experimental Proof of the Law of Error.—** The same point was brought out experimentally in a series of ieseaiches by Prof C S Peirce, of the U. S Coast Survey * He employed a young man, who had had no previous experience whatever in observing, to answer a signal consisting of a sharp sound like a rap, the answer being made upon a telegraph operator's key nicely adjusted Five hundred observations were made on each of twenty-four days The results for the first and last days are plotted below. In the

Fig 6

0⁸ 35   0⁸ 40   0⁹ 45   0ˣ 50   0⁹ 55

Fig 7

0⁸ 20   0⁸ 25   0⁸ 30
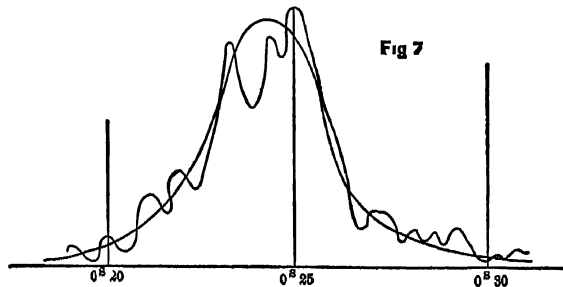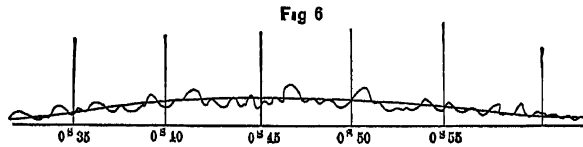
* *Coast Survey Report*, 1870, appendix 21

figures the abscissas represent the interval of time between the signal and the answer, the ordinates the number of observations. The curve is a mean curve for every day, drawn by eye so as to eliminate irregularities entirely   After the first two or three days the curve differed very little from that derived from the theory of least squares   On the first day, when the observer was entirely inexperienced, the observations scattered to such an extent that the curve had to be drawn on a different scale from that of the other days.

38 **General Conclusions.**—On the whole, though we cannot say that the formula $-\dfrac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}$ will truly represent the law of error in any given series of observations, we can say that it is a close approximation.

When in a series of observations we have exhausted all of our resources in finding the corrections, and have applied them to the measured values, the residuum of error may fairly be supposed to have arisen from many sources ; and we conclude from the foregoing investigations that, of any one single law, the best to which we can consider the residual errors subject, and the best to be applied to a set of observations not yet made, is the exponential law of error

The general theorem of Art 17 may therefore be applied to a limited series and be written   *If the observed values of a quantity are of different quality, the most probable value is found by dividing each residual error by the m. s e and making the sum of the squares of the quotients a minimum, if of the same quality, the most probable value is the arithmetic mean of the observed values*

If a set of observations shows a marked divergence from this law a rigid examination will reveal the necessity, in general, of applying some hitherto unknown correction. Thus in the earlier differential comparisons of the compensating base-apparatus of the United States Lake Survey with the standard bar packed in ice, the observed differences did not follow the law of error, as it was fair to suppose that they should, the bars being compensating. There was in-

stead a regular daily cycle some one source of error so far
exceeded the others that it overshadowed them.   A study
of the results was made, and the law of daily change dis-
covered, which gave a means of applying a further correc-
tion   The work done later, after taking account of the
new correction, showed nothing unusual.

Classification of Observations.

39 For purposes of reduction observations may be di
vided into two classes those which are independent, being
subject to no conditions except those fixed by the observa
tions themselves, and those which are subject to certain
conditions outside of the observations, as well as to the con
ditions fixed by the observations.   In the former class, be
fore the observations are made, any one assumed set of
values is as likely as any other; in the latter no set of values
can be assumed to satisfy approximately the observation
equations which does not exactly satisfy the *a priori* condi
tions

For example, suppose that at a station c the angles
AOB, AOC are measured. If the measures of each angle
are independent of those of the other, the angles are found
directly.

The angle BOC could be determined from the relation

$$AOC = AOB + BOC$$

The unknown in this case may be said to be observed in
directly, and therefore independent observations may be
classed as *direct* and *indirect*   The former class is a special
class of the latter.

But if the angle BOC is observed directly as well as
AOB, AOC, then these angles are no longer independent,
but are subject to the condition that when adjusted

$$AOC = AOB + BOC$$

and no set of values can be assumed as possible which does
not exactly satisfy this condition.

10

The observations in this case are said to be *conditioned*
Though we have, therefore, strictly speaking, only two
classes of observations, we shall, for simplicity, divide the
first into two and consider in order the adjustment of

(1) Direct observations of one unknown

(2) Indirect observations of several independent un-
knowns

(3) Condition observations

CHAPTER III.

In the application of the ideal formula of Chapter II, to an actual series of observations, we shall begin with a single quantity which has been directly observed. We shall consider two cases: first, when all of the observed values are of equal quality, and, next, when they are not all of equal quality.

A. Observed Values of Equal Quality.

40. The Most Probable Value; the Arithmetic Mean. We have seen that in a case of directly observed values $H_1, H_2, \ldots, H_n$ of equal quality the most probable value V of the observed quantity is found by taking the arithmetic mean of the values; that is,

$$ V = \frac{\Sigma H}{n} \qquad (1) $$

It has also been shown that the same result will follow by making the sum of the squares of the residual errors a minimum. Thus the observations give the equations,

$$ V = H_1 $$
$$ V = H_2 \qquad (2) $$
$$ V = H_n $$

and V is to be found from

$$ \Sigma (V - H)^2 = a \ min. \qquad (3) $$

that is, from

$$ (V - H_1)^2 + (V - H_2)^2 + \ldots + (V - H_n)^2 = a \ min. \quad (4) $$

By differentiation of (4)

$$(V - M_1) + (V - M_2) + \quad \cdot \quad + (V - M_n) = 0 \quad (5)$$

and
$$V = \frac{[M]}{n} \quad (6)$$

In practice it would evidently be simpler to find the value of the unknown by taking the arithmetic mean of the observed values directly rather than to form the observation equations and find it by making the sum of the squares of the residuals a minimum

It is useful to notice, for purposes of checking, that Eq. (5) may be written

$$[v] = 0 \quad (7)$$

41 As the observed values M are often numerically large and not widely different, the arithmetical work of finding the mean may be shortened as follows

A cursory examination of the observations will show about what the mean value V must be   Let X' denote this approximate value of V, which may conveniently be taken some round number   Subtract X' from each of the observed values $M_1, M_2, \quad \cdot \quad M_n$ in succession, and call the differences $l_1, l_2, \cdot \quad \cdot \ l_n$ respectively.   Then

$$M_1 - X' = l_1$$
$$M_2 - X' = l_2 \quad (1)$$
$$\cdot$$
$$M_n - X' = l_n$$

By addition,

$$[M] - nX' = [l]$$
$$\cdot \quad V = \frac{[M]}{n}$$
$$= X' + \frac{[l]}{n}$$
$$= X' + x' \text{ suppose} \quad (2)$$

Hence all that we have to do is to take the mean x' of the small quantities $l_1, l_2, \qquad l_n$, and add the assumed value X' to the result.

that ... the ... mean ... are in the degrees ...

4. Control of the Arithmetic Mean. In least
squares ... in all computations, it is important to have a
check on or control of the numerical work. This is specially
... ... when a computation takes several weeks, or it
may a month, to complete it. In long computations it is
better for two computers to work together, using different
method whenever possible, and to compare results at in-
tervals. But even this is not an absolute safeguard against
mistakes, for it sometimes happens that both make the same
... for example, writing 6 for 9 or the error. Hence,
even if the computation is made in duplicate, it is advisable
to cast through an independent check which may be re-
ferred to now and again. In computations not duplicated a
control is essential.

A control of the accuracy of the arithmetic mean of a set
of observed values of the same quantity is afforded by the
relation

that is, that the sum of the positive residuals should be
equal to the sum of the negative residuals.

If, however, in taking the arithmetic mean, the sum [V]
of the observed quantities was not exactly divisible by their
number n, the sum of the positive and negative residuals
would not be equal, but the amount of the discrepancy
could easily be estimated and allowed for. For if the value
of the mean taken were too large by a certain amount, the
positive residuals would each be too large, and the negative
residual too large, by that amount. Hence the dis-
crepancy to be expected would be n times the amount that
the approximate quotient taken as the mean differed from
the exact quotient.

Ex In the telegraphic determination of the difference of longitude between St Paul and Duluth, Minn, June 15, 1871, the following were the corrections found for chronometer Bond No 176 at 15h 51m sidereal time from the observations of 21 time stars   (*Report Chief of Engineers U S A*, 1871)

M	v		vv
s	s		
− 8 78	+ 0 04		0 0016
76	+ 02		4
85	+ 11		121
78	+ 04	s	16
51		− 0 23	529
64		− 10	100
68		− 06	36
63		− 11	121
58		− 16	256
80	+ 06		36
75	+ 01		1
78	+ 04		16
96	+ 22		484
64		− 10	100
65		− 09	81
83	+ 09		81
70		− 04	16
64		− 0 10	100
79	+ 05		25
90	+ 16		256
− 8 93	+ 0 19		0 0361
Mean − 8 74	+ 1 03 − 0 99		[vv] = 0 2756
	[v = 2 02		

Taking the observations as of equal precision, we find the arithmetic mean to be − 8 74   This is the most probable value of the correction

The residuals *v* are found by subtracting each observed value from the most probable value according to the relation

$$V - M = v$$

They are written in two columns for convenience in applying the check

$$[v] = 0$$

The exact mean being − 8 74$\frac{4}{21}$, the quantity $\frac{4}{21} \times 21 = 4$ should be, as it is, the numerical difference between the + and − residuals   Hence we may consider the mean to be correctly found

Precision of the Arithmetic Mean.—The degree of confidence to be placed in the most probable value of the unknown is shown by its mean square or probable error.

43 (a) *Bessel's Formula*

If we knew the true value T of the unknown, and consequently the true errors $\varDelta_1, \varDelta_2$ we should have, as in Art. 20, for the m s e of an observation,

$$\mu^2 = \frac{[\varDelta\varDelta]}{n}$$

But we have only the most probable value V and the residual errors $v_1, v_2,$ v_n instead of the true values $T, \varDelta_1, \varDelta_2$ $\varDelta_n$ Now,

$$V - v_1 = M_1 = T - \varDelta_1$$
$$V - v_2 = M_2 = T - \varDelta_2$$
$$V - v_n = M_n = T - \varDelta_n \tag{1}$$

By addition, remembering that $[v] = 0$,

$$nV = nT - [\varDelta] \tag{2}$$

Substitute for V in equations (1) and

$$nv_1 = (n-1)\varDelta_1 - \varDelta_2 -$$
$$nv_2 = \qquad -\varDelta_1 + (n-1)\varDelta_2 - \qquad .$$

Squaring,

$$n^2 v_1^2 = (n-1)^2\varDelta_1^2 + \qquad \varDelta_2^2 + \qquad - 2(n-1)\varDelta_1\varDelta_2 \dots$$
$$n^2 v_2^2 = \qquad \varDelta_1^2 + (n-1)^2\varDelta_2^2 + \qquad - 2(n-1)\varDelta_1\varDelta_2 \dots$$

By addition, assuming that the double products destroy each other, positive and negative errors being equally probable,

$$n^2[vv] = \{(n-1)^2 + (n-1)\}[\varDelta\varDelta]$$
$$[vv] = \frac{n-1}{n}[\varDelta\varDelta]$$
$$= (n-1)\mu^2$$

and

$$\mu^2 = \frac{[vv]}{n-1} \tag{3}$$

which gives the m s e of an observation.

Now, from Art. 19,

$$\mu_o = \frac{\mu}{\sqrt{n}}$$

$$\therefore \mu_o = \sqrt{\frac{[vv']}{n(n-1)}}$$

which gives the m s e of the arithmetic mean of n observations of equal precision

The result, $\mu^2 = \frac{[vv]}{n-1}$, might have been inferred *à priori*. For the series of residuals v_1, v_2 . found from the arithmetic mean V of the observed values approximates closely to the true series of errors Δ from which the law of error was derived   Hence we conclude that the formula

$$\mu^2 = \frac{[vv']}{n} \qquad (4)$$

would be a close approximation to the m s e of an observation   It is, however, not satisfactory, from the fact that it ought to become indeterminate when $n = 1$, which it does not   For when $n = 1, v = 0$, and unless the denominator of (4) is equal to o, μ would be equal to o, that is, the first observation would give the true value of the unknown, which is absurd   Hence we should expect the formula to be of the form

$$\mu^2 = \frac{[vv']}{n-1}$$

which becomes of the indeterminate form $\frac{0}{0}$ when $n = 1$

44 As in Art 23, we may show from the expansion of $\left(\frac{[vv]}{n-1} - \mu^2 \right)^2$ that the square of the m s e. of $\mu^2 = \frac{[vv']}{n-1}$ is equal to $\mu^2 \sqrt{\frac{2}{n-1}}$. We have, therefore,

$$\mu^2 = \frac{[vv]}{n-1}\left(1 \pm \sqrt{\frac{2}{n-1}} \right)$$

and when n is very large,

$$r = \frac{?}{\sqrt{n}} \left(1 + \ldots \right)$$

that is, the mean uncertainty of r is

$$\frac{r}{\sqrt{2n}}$$

45. From the constant relation existing between the m.s.e. and p.e. given in Art. ... we have for the p.e. of an observation and of the arithmetic mean of n observations respectively,

$$r = \rho \cdot \varphi$$

$$\rho = \frac{?}{\sqrt{n}}$$

$$r = \rho = \frac{?}{\sqrt{n}}$$

where

$$\rho = \ldots 0.6745 \ldots m.s.e.$$

46. If we consider the m.s.e. ρ of the arithmetic mean of n observations as the true error of the mean l, the results, $\rho = \frac{?}{\sqrt{n}}$ and $\rho = \frac{?}{\sqrt{n}}$ may be derived very neatly as follows:

We have

$$l = v_1 = \Delta$$
$$l = v_2 = \Delta$$
$$l = v_n = \Delta_n$$

By addition, taking the mean

$$l = \frac{1}{n}(v_1 + v_2 + \ldots + v_n) + \frac{1}{n}(\Delta_1 + \Delta_2 + \ldots + \Delta_n)$$

$$l = \frac{1}{n}(\Delta_1 + \Delta_2 + \ldots + \Delta_n)$$

11

$$\text{Error of } V = \frac{1}{n}(\varDelta_1 + \varDelta_2 + \quad + \varDelta_n)$$

and

$$\mu_o^2 = \frac{1}{n^2}(\varDelta_1 + \varDelta_2 + \quad + \varDelta_n)^2$$

$$= \frac{[\varDelta\varDelta]}{n^2}$$

But from Art 20

$$\mu^2 = \frac{[\varDelta\varDelta]}{n}$$

$$\mu_o = \frac{\mu}{\sqrt{n}}$$

Again,

$$\varDelta = T - M$$

$$= V \pm \frac{\mu}{\sqrt{n}} - M$$

$$= v \pm \frac{\mu}{\sqrt{n}}$$

and

$$\mu^2 = \frac{[\varDelta\varDelta]}{n}$$

$$= \frac{1}{n}\left[\left(v \pm \frac{\mu}{\sqrt{n}}\right)^2\right]$$

Hence, remembering that $[v] = 0$,

$$\mu^2 = \frac{[vv]}{n - 1}$$

47 (b) *Peters' Formula*

The m s e and p e of a series of observed values may he more rapidly computed from the sum of the errors rather than from the sum of their squares by means of the convenient formula first given by Dr Peters *

From the equation

$$[vv] = \frac{n - 1}{n}\,[\varDelta\varDelta]$$

we have approximately, without regard to sign,

$$\{ \quad \}$$

$$\{ \quad \}$$

Adding and dividing by n,

$$n \quad \sqrt{\quad} \quad n$$

But from Art. ··

$$\}$$

$$\{ \quad \}$$

$$\text{nearly.}$$

For a demonstration of this formula more rigorous in form see *Astron. Nachr.*, No. ····

As in Art. ··, it follows that the precision of this formula is expressed by the complete form

$$\}$$

the last term giving the mean uncertainty of ρ.

For the p. e. of an observation and of the arithmetic mean of n observations we have respectively

$$r \quad \rho \pm \rho$$

$$r_n \quad n$$

The mode of deriving Peters' formula given in the preceding is approximate, and the formula itself is not very

close for small values of n Thus when $n = 2$, if d denotes the difference of the observed values, then by Bessel's formula $\mu = \dfrac{d}{\sqrt{2}}$, and by Peters' formula $\mu = 1\ 25\ \dfrac{d}{\sqrt{2}}$, which is one-fourth greater   The corresponding probable errors are in the same ratio

A formula for the probable error of the mean which answers better than Peters' for small values of n has been derived by Fechner (Poggendorff, *Annalen*, *Jubelband*, 1874) as follows

$$r_o = \frac{1\ 1955}{\sqrt{2n - 0\ 8548}}\ \frac{[v}{n}$$

As this formula is troublesome to compute, and as it gives results agreeing closely with those found from Peters' formula when n is a moderately large number, there is no advantage to be derived from the use of it in ordinary work

48 Collecting the formulas for finding the p e of a single observation and of the arithmetic mean of n observations, we have

$$r = 0\ 6745\ \sqrt{\frac{[vv]}{n-1}} \qquad r = 0\ 8453\ \frac{[v}{\sqrt{n(n-1)}}$$

$$r_o = 0\ 6745\ \sqrt{\frac{[vv]}{n(n-1)}} \qquad r_o = 0\ 8453\ \frac{[v}{n\sqrt{n-1}}$$

To save labor in the numerical work I have computed tables containing the values of the coefficients of $\sqrt{[vv]}$ and $[v$ in these equations for values of n from 2 to 100   (See Appendix, Tables II , III)

If Bessel's formula is used compute first $[vv]$, then $\sqrt{[vv]}$ can be taken from a table of squares closely enough. This square-root number multiplied by the number in Table II corresponding to the given value of n gives the p e sought It Peters' formula is used multiply the sum of the residuals, without regard to sign, by the numbers in Table III corresponding to the argument n

Control of ... ... affected by the derivation ... ... observed value ... and the arithmetic mean of ratio

We have

Square and add,

$$ ... $$

The value of ... is to found from a table of squares or ... If the numbers ... are large, an arithmometer ... ... by multiplying and dividing, may be ... with ... ...

... ... the ... in Art. ... ... the value of F, W, ... W_n ... that with ... ... after a single reduction,

$$ ... $$

Approximate Method of Finding the Precision. ... ... the precision of a single observation ... the ... ... the ... ... the principle proved in Art. ... ... the actual ... ... and for ... the precision ... ... If, then, we find from the observation of ... ... and ... ..., we may assert that the ... ... more than ... The precision ... ... the ... only ...

The same principle enables us to estimate roughly the precision ... ... of observations. A glance at the

measured results will show the largest and smallest, and their difference may be taken as the range in the results, and half the difference as the maximum error   Hence, since in an ordinary series of from 25 to 100 observations the maximum error may be expected to be from 3 to 4 times the p e , we may take *the p e to be from ⅛ to ⅙ of the range of the errors of observation*

This result may be confirmed as follows   Expanding the exponential function $\varphi(\varDelta)$ in a series, we may write

$$\varphi(\varDelta) = P - Q\varDelta^2 + R\varDelta^4 - . \quad .$$

where $P, Q, R,$ are constants

If a denotes the maximum error, then, since the probability of the occurrence of an error between the limits $+a$ and $-a$ is certainty, we have, taking two terms of the series,

$$\int_{-a}^{+a} (P - Q\varDelta^2)\, d\varDelta = 1$$

Also, since a is the maximum error, its probability is zero

$$\cdot \quad P - Qa^2 = 0$$

From these two equations P and Q may be found in terms of a

To find the p e r we have by definition (see Art 21)

$$\int_{-r}^{+r} (P - Q\varDelta^2)\, d\varDelta = \frac{1}{2}$$

or

$$Pr - \frac{1}{3} Qr^3 = \frac{1}{2}$$

Substituting for P and Q their values, and solving for r, we find

$$r = \frac{a}{3} \text{ nearly}$$

that is, *the p e is approximately ⅓ of the maximum error, or ⅙ of the range of the errors of observation.*

A closer approximation would be found by taking three terms of the series for $\varphi(\Delta)$ We should then find

$$r = \frac{a}{4} \text{ nearly}$$

See note by Capt Basevi, R E , in *G T Survey of India*, vol iv , also Helmert in *Zeitschr fur Vermess* , vol vi

Ex We shall now apply the preceding formulas to the example in Art 42 to find the m s e and p e of the arithmetic mean and of a single observation

(1) *The m s e and p e of the arithmetic mean*
These we may find in two ways
(a) From the sum of the squares of the residuals (Art 43)

$$\mu_0 = \sqrt{\frac{[vv]}{n(n-1)}}$$

$$= \sqrt{\frac{0\ 2756}{21 \times 20}}$$

$$= 0\ 026$$

$$r_0 = 0\ 6745 \times 0\ 026$$

$$= 0\ 017$$

or from Table II at once

$$r_0 = 0\ 525 \times 0\ 033$$

$$= 0\ 017$$

(b) From the sum $[v$ of the residuals (Art 47)
The multiplier in Table III corresponding to the number 21 is 0 009

$$r_0 = 2\ 02 \times 0\ 009$$

$$= 0\ 018$$

(2) *The p e of a single observation*
From Tables II and III directly

$$r = 0\ 525 \times 0\ 151 = 0\ 079$$

$$r = 2\ 02\ \times 0\ 041 = 0\ 082$$

Check (α) Let the residuals be arranged in order of magnitude They are

0 23 0 22 0 19 0 16 0 16 0 11 0 11 0 10 0 10 0 10 0 09
0 09 0 06 0 06 0 05 0 04 0 04 0 04 0 04 0 02 0 01

The residual 0 09 occupies the middle place, and is therefore the p e r of a single result (Art 21) The computation above gives 0 08
Check (β) See Art 50

$$\text{Range} = 0\ 22 + 0\ 23 = 0\ 45$$

$$r = \frac{0\ 45}{6} = 0\ 08$$

The values found by the different methods agree reasonably well

51 **The Law of Error Tested by Experience.**—We shall now test our example and see how closely it conforms to the law of error, and hence be in a better position to judge of how far the law of error itself is applicable in practice This is the *à posteriori* proof intimated in Art. 11 as necessary for the demonstration of the law.

(1) The number of + residuals is 12, and the number of — residuals is 9

(2) The sum of the + residuals is 1 03, and the sum of the — residuals is 0 99

(3) The sum of the squares of the + residuals is 1417, and of the — residuals is 1339

(4) The p. e of a single observation is 0 08.   To find the number of observations we should expect whose residual errors are not greater than 0 10, we enter Table I with the argument $\frac{0\ 10}{0\ 08} = 1\ 25$ and find 0 60   This multiplied by 21 gives 13 as the number of errors to be expected not greater than 0 10   By actual count we find the number observed to be 14

To find the number to be expected between 0 10 and 0 20 we enter the table with the argument $\frac{20}{08} = 2\ 50$ and find 0 91. From this deduct 0 60 and multiply the remainder by 21. This gives 6   The number observed is 5.

The number to be expected over 0 20 is, by theory, 2. The number observed is 2

The preceding results are collected in the following table

Limits of Error	Number of Errors	
	Theory	Observation
s s 0 00 to 0 10	13	14
0 10 to 0 20	6	5
over 0 20	2	2

The agreement, especially for the larger errors, is not very close. Prof Safford, to whom I referred this example, states that in a new reduction of Bradley's observations made by, Auwers a much better agreement between experience and theory is found Three sources of systematic error enter into the observations which were not taken into account in Bessel's reduction

(a) Personal equation Bradley was not the only observer, Mason and Green being the others.

(b) Local deviation of the plumb-line

(c) The assumption that the collimation of Bradley's Quadrant did not vary irregularly

Auwers has in a great degree overcome these difficulties in his reduction He finds, for example, the number of errors over 4″ to be 2 instead of 6, thus agreeing with the theoretical number

For another illustration see Helmert's discussion of the errors in Koppe's triangulation for determining the axis of the St Gothard Tunnel. (*Zeitschr für Vermess.*, vol. v. pp 146 seq)

52 As the Gaussian law of error is found to apply reasonably well to other phenomena, such, for example, as statistical questions, guesses, etc , it has been often rashly assumed to be of universal application , and when prediction and experience are found not to agree, the validity of the law in any case has been as rashly impugned

In the fundamental investigation in Art 15 the hypotheses there made are satisfied by other functions of the measured values besides the arithmetic mean Thus, for example, taking the geometric mean g, the resulting law of error would be of the form

$$\varphi\left(\frac{\varDelta}{g}\right) \equiv ce^{-h^2 \log^2 \frac{\varDelta}{g}}$$

where c and h are constants

This form appears to apply to many statistical questions [*] better than the Gaussian law That it does so is, how-

A Caution as to the Application of the Tests of Precision.

the lines are run in the same direction, it is to be expected that the final result obtained from measurements in opposite directions will be nearer the truth   The conclusion arrived at by trusting to the m s e alone would be illusory

Again, in measuring a horizontal angle, if the same part of the limb of the instrument is used in making the readings the results may be very accordant and the m s e consequently small, but the angle itself may not be anywhere near the truth   This would be shown, for example, by the large discrepancies in the sums of angles of triangles measured in this way from the theoretical sums   For a long time this contradiction was a source of much perplexity, and many good instruments were most unjustly condemned *   At last the discovery was made that it was mainly owing to periodic errors of graduation of the limb which, when corrected for, made the remaining errors fairly subject to the law of error   This most important discovery may be said to have revolutionized the art of measuring horizontal angles

The difficulty may be explained in this way   In the derivation of the m s e from a series of n observed quantities $M_1, M_2,$ we had the observation equations

$$V - M_1 = v_1$$
$$V - M_2 = v_2$$

$$V - M_n = v_n$$

Also $$\mu^2 = \frac{[vv]}{n - 1}$$

Now, if we suppose each of the observed quantities to be changed by the same amount c, which may be of the nature of a constant error or correction, so that they become $M_1 + c, M_2 + c,$ the most probable value, instead of being V, will be $V + c$ Also since

$$v = (V + c) - (M + c)$$
$$= V - M$$

the residual errors will be the same as before

* See, for example, *G T Survey of India*, vol ii pp 51, 96

Hence μ^2 is unchanged, and we see, therefore, that the m s e makes no allowance for constant errors or corrections to the observed quantity These are supposed to be eliminated or corrected for before the most probable value and its precision are sought

Another common misapprehension is the following From Arts 19 or 46 the relation between the m s e of a single observation μ and the m s e of the mean of n observations μ_0 is

$$\mu_0 = \frac{\mu}{\sqrt{n}}$$

This formula shows that by repeating the measurement a sufficient number of times we can make the m s e of the final result as small as we please Nothing would, therefore, seem to be in the way of our getting an exact result, and that we could do as good work with a rude or imperfect instrument as with a good one by sufficiently increasing the number of observations

Experience, however, shows that in a long series of measurements we are never certain that our result is nearer the truth than the smallest quantity the instrument will measure If an instrument measures seconds we cannot be sure that by repeating the observations we can get the nearest hundredth or tenth In a word, we cannot measure what we cannot see

Take an example With the meridian circle P of Rogers found the p e of a single complete observation in declination to be $\pm$ 0″ 36, and the p e of a single complete observation in right ascension for an equatorial star to be $\pm$ 0ˢ 026. He says "If, therefore, the p e can be taken as a measure of the accuracy of the observations, there ought to be no difficulty in obtaining from a moderate number of observations the right ascension within 0ˢ 02 and the declination within 0″ 2 Yet it is doubtful, after continuous observations in all parts of the world for more than a century, if there is a single star in the heavens whose absolute co-ordinates are

known within these limits "[*] The explanation is, as inti-
mated, that constant errors are not eliminated by increasing
the number of observations Accidental errors are elimi-
nated by so doing , and if a number of observations ex-
pressed by an infinity of a sufficiently high order could be
taken, so that the constant errors entering in the different
series could be classed as accidental, these errors would
mutually balance in the reduction and we should arrive at
the true result

Closely allied to the preceding is the common idea that
if we have a poor set of observations good results can be
derived from them by adjusting them according to the
method of least squares, or that if work has been coarsely
done such an adjustment will bring out results of a higher
grade   A seeming accuracy is obtained in this way, but it
is a very misleading one   The method of least squares is
no philosopher's stone it has no power to evolve reliable
results from inferior work

A third source of uncertainty from the same cause may
be mentioned   It may happen that the value obtained of
the p e is numerically greater than that of the observed
quantity itself It is then a question whether in subsequent
investigations we should use the value of the observed
quantity as found or neglect it   This depends on circum-
stances   It is ever a principle in least squares to make use
of all the knowledge on hand of the point at issue   If we
have strong *à priori* reasons for expecting the value zero it
would be better to take this value   Thus if we ran a line
of levels between two points on the surface of a lake we
should expect the difference of height to be zero   If the
p e of the result found were greater than the result itself
it would be allowable in this case to reject the determina-
tion   On the other hand, when we have no *à priori* know-
ledge, as in determinations of stellar parallax, for example,
if the p e of the value found were in excess of the value
itself, as is sometimes the case,[†] we could do nothing but

* *Proc Amer Acad Sci* , 1878, p 174   † See, for example, Newcomb, *Astronomy*, app vii

take the value resulting from the observations, unless, indeed, it came out with a negative sign, and then its unreliable character would be evident

54 **Constant Error.**—The remarks on constant error in the preceding articles lead us to notice an example or two of the detection and treatment of this great bugbear of observation

We suspect the presence of constant errors in a series of observations from the large range in the results—a range greater than would naturally be expected after all known corrections have been applied   Great caution is necessary in dealing with such cases, and one should be in no hurry to jump at conclusions

Sometimes the sources of error are detected without much trouble' Thus in measuring an angle with a theodolite, if the instrument is placed on a stone pillar firmly embedded in the ground, the range in results, if targets are the signals pointed at, would not usually be over 10″ in primary work, and on reading to a number of signals in order round the horizon the final reading on closing the horizon would be nearly the same as the initial reading on the same signal   If next the instrument were placed on a wooden post or tripod, and readings made to signals in order round the horizon in the same way as before, the final reading might differ from the initial by a large amount. The observations might also show that the longer the time taken in going around the greater the resulting discrepancy. The natural inference would be that in some way the wooden post had to do with the discrepancy in the results. In an actual case* of this kind examination showed the change to be most uniform on a day when the sun shone brightly   Measurements were then made at night, using

lamps as signals on the distant stations, and the same change was observed, only it was in the opposite direction

The effect on the value of an angle of this twist of station, assuming it to act uniformly in the same direction during the time of observation, can be eliminated by the method of observation first reading to the signals in one direction and then immediately in the opposite direction, and calling the mean of the difference of the two sets of readings a single value of the angle So also in azimuth work the mean of the difference of the readings, star to mark, and mark to star, gives a single value free from station-twist

This mode of procedure is in accordance with the general principle to eliminate a correction, when possible, by the method of observation, rather than to compute and apply it See Art 2

The effort to avoid systematic error causes in general a considerable increase of labor, and sometimes this is very marked. For example, in the micrometric comparison of two line measures belonging to the U S Engineers the results found by different observers showed large discrepancies The micrometer microscopes used were of low power, with a range of about one mm between the upper and lower limits of distinct vision Examination showed that the discrepancies arose mainly from focusing, each observer's results being tolerably constant for his own focus As the value of a revolution of the micrometer screw entered into the reduction of the comparison work, and as this value was obtained from readings on a space of known value, error of focusing entered from this source Hence a value of the screw had to be determined from a special set of readings taken at each adjustment, and this value used in reducing the regular observations made with the same focus Had the microscopes been of high power it would have been sufficient to have determined the value of the screw once for all, since the error arising from change of focus could have been classed as accidental

In trying to avoid systematic error the observer will, as he gains in experience, take precautions which would at first seem to be almost childish Good work can only be had at the cost of eternal vigilance

55 **Necessary Closeness of Computation.**—The number of observations necessary to the proper determination of a quantity may be approximated to by referring to the m s e of these observations If, after planning the method of observing with the view of eliminating "constant error," we find that increasing the number of observations beyond a certain limit does not sensibly affect the m s e , we may conclude that we have a sufficient number It is evident that increasing or decreasing this number will affect the result more or less But we cannot say that we are nearer the truth in either case

Hence the folly of a too rigorous computation An approximate value being the best that we can have at any rate, no weight is added to it by carrying out that value to a great many decimal places Thus if the most probable value of a quantity computed in an approximate way is V, whereas the value found from the same observations by a rigorous computation is $V + c$, we may estimate the allowable value of c as follows Let $\varDelta_1, \varDelta_2, \quad \varDelta_n$ be the errors of V, then $\varDelta_1 + c, \varDelta_2 + c, \quad \varDelta_n + c$ are the errors of $V + c$ Hence

$$\mu_{v^2 + c} = \frac{1}{n}\left\{ (\varDelta_1 + c)^2 + (\varDelta_2 + c)^2 + \quad + (\varDelta_n + c)^2 \right\}$$

$$= \frac{[\varDelta\varDelta]}{n} + c^2$$

$$= \mu_v^2 + c^2$$

and

$$\mu_{v + c} = \mu_v \left(1 + \frac{1}{2}\frac{c^2}{\mu_v^2} \right) \text{ approximately.}$$

Now, we may safely allow the difference between $\mu_{v + c}$ and

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μ_V to be $\dfrac{\mu_V}{100}$ Hence theie will be no appreciable eiroi in-
troduced by computing in such a way that

$$\frac{1}{2}\frac{c^2}{\mu_V}, < \frac{1}{100}$$

that is, when the error c committed is, roughly, $\frac{1}{10}$ of the
m s e

B *Observed Values of Different Quality*

**56 The Most Probable Value: the Weighted
Mean.**—It has been shown in Art 17 that if the directly
observed values M_1, M_2, M_n of a quantity are of dif-
ferent quality, the most probable value is found by multiply-
ing each residual erroi of observation by the ieciprocal of
its m s e, and making the sum of the squares of the pro-
ducts a minimum, that is, with the usual notation,

$$\frac{v_1^{\,2}}{\mu_1}, + \frac{v_2^{\,2}}{\mu_2}, +   + \frac{v_n^{\,2}}{\mu_n^{\,2}} = a \text{ min} \tag{1}$$

or

$$\left(\frac{V - M_1}{\mu_1}\right)^2 + \left(\frac{V - M_2}{\mu_2}\right)^2 +   + \left(\frac{V - M_n}{\mu_n}\right)^2 = a \text{ min} \tag{2}$$

By differentiation and reduction,

$$V = \left[\frac{M}{\mu^2}\right] - \left[\frac{1}{\mu^2}\right] \tag{3}$$

We have, therefore, the equivalent rule

*If the observed values of a quantity are of different quality,
the most probable value is found by multiplying each observed
value by the reciprocal of the square of its m s e, and dividing
the sum of the products by the sum of the reciprocals*

The form of the expression for V suggests another
stand-point from which to consider it   Let p_1, p_2, p_n

be the numerical parts of $\frac{1}{\mu_1^2}, \frac{1}{\mu_2^2}, \quad \frac{1}{\mu_n^2}$, such that each is of the type

$$p = \frac{(\text{unit of measure})^2}{\mu^2}$$

then equation (3) may be written

$$V = \frac{[pM]}{[p]} \qquad (4)$$

Also, since $\frac{1}{\mu_1^2}, \frac{1}{\mu_2^2}, \quad \frac{1}{\mu_n^2}$ are similarly involved in the numerator and denominator of the value of V, this value will remain the same if $p_1, p_2, \quad . \ p_n$ are taken any numbers whatever in the same proportion to $\frac{1}{\mu_1^2}, \frac{1}{\mu_2^2}, \quad \frac{1}{\mu_n^2}$, that is, if $p_1, p_2, \quad p_n$ satisfy the relations

$$p_1 = \frac{\mu^2}{\mu_1^2}, p_2 = \frac{\mu^2}{\mu_2^2}, \quad , p_n^2 = \frac{\mu^2}{\mu_n^2} \qquad (5)$$

where μ is an arbitrary constant

57 Let us look at this question from another point of view It is in accordance with our fundamental assumptions that observations of a quantity made under the same conditions, so that there is no *à priori* reason for choosing one before another, are of the same quality They require the same expenditure of time, labor, money, etc If, therefore, we represent the quality of a single observation of a certain series by unity, the quality of the arithmetic mean of p such observations, as it would require p times the expenditure to attain it, would be represented by p Let us suppose, then, that the arithmetic mean of p_1 observed values of a certain quality is M_1, of p_2 other observed values of the same quality it is M_2, and so on The total number of observed values is $[p]$ All of the observed values being of the same quality, the most probable value

V of the unknown is given by the arithmetic mean, that is, by

$$V = \frac{\text{sum of the values of all the sets}}{\text{sum of number of obs in each set}}$$

$$= \frac{p_1 M_1 + p_2 M_2 + \quad + p_n M_n}{p_1 + p_2 + \quad + p_n}$$

$$= \frac{[pM]}{[p]}$$

which is of the same form as Eq 4, above.

The numbers $p_1, p_2, \quad p_n$ are called the *weights*, or, better, the *combining weights*, of the observed values, and the mean value V is called the *weighted mean*

In view of this definition, the general principle stated in Art 56 may be replaced by the following necessarily equivalent one

If the observed values of a quantity are of different weights, the most probable value is found by multiplying the square of each residual error of observation by its weight, and making the sum of the products a minimum

Thus the most probable value V is found from

$$[pvv] = \text{a min.}$$

that is, from

$$p_1(V - M_1)^2 + p_2(V - M_2)^2 + \quad + p_n(V - M_n)^2 = \text{a min.}$$

By differentiation,

$$p_1(V - M_1) + p_2(V - M_2) + \quad + p_n(V - M_n) = 0$$

whence

$$V = \frac{[pM]}{[p]}$$

This form of the value of V leads to the rule:

If the observed values of a quantity are of different weights, the most probable value is found by multiplying each observed value by its weight, and dividing the sum of the products by the sum of the weights

As in the case of the arithmetic mean (Art 40), it is evidently simpler in practice to find the weighted mean directly by this rule rather than from the minimum equation

If the observed values M are numerically large we may lighten the arithmetical work by finding V by the method of Art 41 Proceeding as there indicated, we have

$$V = X' + \frac{[pl]}{[p]}$$
$$= X' + x'' \text{ suppose}$$

58 Reduction of Observed Values to a Common Standard.—The principle of the weighted mean is evidently an extension of that of the arithmetic mean, as was pointed out long ago by Cotes, Simpson, and others It merely amounts to finding a mean of several series of means, the unit of measure being the same in each As soon, therefore, as results of different weights are changed into others having a common standard of weight, the rules for combining and finding the precision of observed quantities of the same weight can be applied to weighted quantities

This change we are enabled to make by means of the relation (5), Art 56, which may be written

$$\mu_1 = \frac{\mu}{\sqrt{p_1}}, \; \mu_2 = \frac{\mu}{\sqrt{p_2}}, \qquad \mu_n = \frac{\mu}{\sqrt{p_n}}$$

Now, since $\mu_1, \mu_2,$ μ_n are the m s e of $M_1, M_2,$ $. M_n,$ the m s e of $M_1\sqrt{p_1}, M_2\sqrt{p_2},$ $M_n\sqrt{p_n}$ would each be the same quantity μ

Hence if a series of observed values M_1, M_2 M_n have the weights $p_1, p_2,$ $p_n,$ they are reduced to the same standard by multiplying by $\sqrt{p_1}, \sqrt{p_2},$ $\sqrt{p_n}$ respectively

For example, given the observation equations,

$$V - M_1 = v_1 \text{ weight } p_1$$
$$V - M_2 = v_2 \quad `` \quad p_2$$

$$V - M_n = v_n \quad `` \quad p_n$$

to find the most probable value of V

Reducing to the same standard of weight, we have the equations

$$\sqrt{p_1}\, V - \sqrt{p_1}\, M_1 = \sqrt{p_1}\, v_1$$
$$\sqrt{p_2}\, V - \sqrt{p_2}\, M_2 = \sqrt{p_2}\, v_2$$

$$\sqrt{p_n}\, V - \sqrt{p_n} M_n = \sqrt{p_n} v_n$$

and the most probable value of V is found by making

$$(\sqrt{p_1}\, v_1)^2 + (\sqrt{p_2}\, v_2)^2 + \quad + (\sqrt{p_n}\, v_n)^2 = \text{a min.}$$

that is, by making

$$(\sqrt{p_1}\, V - \sqrt{p_1}\, M_1)^2 + (\sqrt{p_2}\, V - \sqrt{p}\, M_2)^2 + {} $$
$$+ (\sqrt{p_n}\, V - \sqrt{p_n}\, M_n)^2 = \text{a min.}$$

Reducing this equation, we find, as before,

$$V = \frac{[pM]}{[p]}$$

59 Computation of the Weights.—If $p_1, p_2, \ldots p_n$ represent a series of weights corresponding to the m s e. $\mu_1, \mu_2, \ldots \mu_n$, then we have the relations

$$p_1 \mu_1^2 = p_2 \mu_2^2 = \quad = p_n \mu_n^2 = \mu^2$$

where the value of μ is entirely arbitrary The combining weights are, therefore, known when the m s e $\mu, \mu_1, \mu_2, \quad \mu_n$ are known

These relations suggest that it would be convenient to define μ as the m s e of a single observation assumed to be of the weight unity

We shall define μ in this way, so that in future it is understood that the *standard* to which observations of different weights are reduced for comparison and combination is the fictitious observation whose weight is unity and whose m s e is μ

60 Control of the Weighted Mean.—Eq. 2, Art 57, may be written

$$[pv] = 0$$

Hence if the weighted mean V has been computed correctly, the sum of the products of each residual error by its weight is equal to zero

Usually, however, V is not an exact quotient—that is, $[pM]$ is not exactly divisible by $[p]$—and then the discrepancy of $[pv]$ from zero is evidently equal to the product of the sum of the weights and the difference between the exact value of V and the approximate value used. See Art 42

Ex 1 —Deduce the relation $[pv] = 0$ from the observation equations directly

[Multiply each observation equation by its weight, and add the products]

Ex 2 —Find the most probable value of the velocity of light from the following determinations by Fizeau and others

$$298000 \text{ kil} \pm 1000 \text{ kil}$$
$$298500 \text{ '' } \pm 1000 \text{ '' }$$
$$299990 \text{ '' } \pm 200 \text{ '' }$$
$$300100 \text{ '' } \pm 1000 \text{ '' }$$
$$299930 \text{ '' } \pm 100 \text{ '' }$$

(*Amer Jour Sci* , vol xix)

The weights, being inversely is the squares of the probable errors, are as the numbers 1, 1, 25, 1, 100   (Art 59)

(a) *Direct solution*

M	p	pM
298000	1	298000
298500	1	298500
299990	25	7499750
300100	1	300100
299930	100	29993000
	128	38389350

$$V = \frac{[pM]}{[p]} = 299917 \text{ kil}\ \text{approx} ,$$

the exact value being $299916\frac{11}{64}$ kil

(b) *Solution according to Art 57*

Assume $X' = 298000$

l	p	pl
0	1	0
500	1	500
1990	25	49750
2100	1	2100
1930	100	193000
	128	245350

$$x'' = \frac{[pl]}{[p]} = 1916\tfrac{51}{64}$$

and $V = 298000 + 1916\tfrac{51}{64} = 299916\tfrac{51}{64}$, as before

Control Take $V = 299917$, and proceed to find $[pv]$ (See Art 60)

v	p	pv
1917	1	1917
1417	1	1417
$-$ 73	25	$-$ 1825
$-$ 183	1	$-$ 183
$-$ 13	100	$-$ 1300
		26

The discrepancy should be

$$128\left(299917 - 299916\tfrac{51}{64}\right) = 26$$

which it is

61 The Precision of the Weighted Mean.—Since the weighted mean V is the arithmetic mean of $[p]$ observations of the unit of weight, its weight is $[p]$. Hence the m s e μ_V of V is found from

$$\mu_V^2 = \frac{\mu^2}{[p]}$$

where μ is the m s e of an observation of the unit of weight (standard observation)

According to Art 58, the value of μ may be found by writing $\sqrt{p_1}\,v_1$, $\sqrt{p_2}\,v_2$, . for v_1, v_2, .. in the formulas derived for observations of the same weight. Hence, substituting in Bessel's and in Peters' formulas Arts 43, 47, we have

$$\mu^2 = \frac{[pvv]}{n-1} \qquad\qquad \mu = 1\ 2533 \frac{\overline{\sqrt{p}\ v}}{\sqrt{n(n-1)}}$$

and therefore

$$\mu_V^2 = \frac{[pvv]}{[p](n-1)} \qquad\qquad \mu_V = 1.2533 \frac{\overline{\sqrt{p}\ v}}{\sqrt{[p]\,n(n-1)}}$$

These expressions reduce to those for the arithmetic mean where the observed values are of the same weight by putting $[p] = np$

62 *Control of* $[pvv]$ —A control of the accuracy of $[pvv]$ is afforded by the derivation of this quantity from the observed values directly

The observation equations are

$$v_1 = V - M_1 \quad \text{weight } p_1$$
$$v_2 = V - M_2 \qquad \text{`` } \quad p_2$$
$$\dot{v_n} = V - M_n \qquad \text{`` } \quad p_n$$

Hence

$$p_1 v_1^2 = p_1 V^2 - 2p_1 V M_1 + p_1 M_1^2$$
$$p_2 v_2^2 = p_2 V^2 - 2p_2 V M_2 + p_2 M_2^2$$

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By addition,

$$[pvv] = [p]V^{2} - 2[pM]V + [pMM]$$

$$= [pMM] - \frac{[pM]^{2}}{[p]}$$

since

$$V = \frac{[pM]}{[p]}$$

In using this formula the troublesome term is $[pMM]$
With large values of M it is better to deduce it from the
column pM, already computed in finding the weighted mean
This is specially advisable if one has a machine for perform-
ing multiplications With small values of M a table of
squares is best

With large values of M we may perhaps proceed still
more conveniently by the method explained in Arts 41, 57
With the same notation and reasoning as there employed it
is found that

$$[pvv] = [pll] - \frac{[pl]^{2}}{[p]}$$

$$= [pll] - [pl]x''$$

where the quantities l are numerically small

Ex The linear values found for the space 0^{m} 00 to 0^{m} 05 of inch $[ab]$
on the standard steel foot I F of the G T Survey of India were as follows
0^{m} 050027, 0^{m} 049971, 0^{m} 050019, 0^{m} 050079, 0^{m} 050021, 0^{m} 050011 The num-
bers of measures in these determinations were 6, 6, 15, 15, 8, 8 respec-
tively

Taking the numbers of measures as the weights of the respective determi-
nations, required the most probable value of the space and its p e
The direct solution presents no difficulty The value of V may be found
as in Ex Art 60, and thence the residuals v The m s e or p e follows
from the formulas of Art 61

We shall give the solution according to the methods of Arts 57 and 62,
which are advantageous in this case on account of the large numbers that
enter

Assume $X' = 0\,049971$

l	p	pl	pll
56	6	336	18816
0	6	0	0
48	15	720	34560
108	15	1620	174960
50	8	400	20000
40	8	320	12800
	58	3396	$261136 = [pll]$
			$198842 = [pl]x''$
			$62294 = [pvv]$

and

$$x'' = \frac{3396}{58} = 59 \qquad\qquad r = 0\,6745\sqrt{\frac{62294}{58(6-1)}}$$

Hence

$V = 0\,049971 + 0\,000059$
$\quad = 0\,050030 \qquad\qquad\qquad = 0\,000010$

$$V = 0^m\,050030 \pm 0^m\,000010$$

By choosing the approximate value X' equal to the smallest of the measures or equal to the greatest of them, all of the remainders l have the same sign, which is a great convenience in computation

In this example an important practical point occurs, and one often overlooked The p e is not computed from the original observations, but from these observations grouped in six sets of means These means we have treated as if original observations of certain weights Had the original observations been accessible we should have used them, and would most probably have found a different value of the p e from that which we have obtained This arises from the small number of observations in the several sets In good work the difference to be expected between the value of the p e found from the means and that found from the original observations would be small Still, whenever there is a choice, the p e should always be deduced

from the original observations rather than from any combinations of them

The weighted mean value V would evidently be the same whether computed from the partial means or from the original observations

Observed Values Multiples of the Unknown

63 Let the observed values M_1, M_2, . M_n be multiples of the same unknown X, that is, be of the form a_1X, a_2X, a_nX, where a_1, a_2, a_n are constants given by theory for each observation. The values $\frac{M_1}{a_1}$, $\frac{M_2}{a_2}$, $\frac{M_n}{a_n}$ of X may be regarded as directly observed values of unequal weight   If μ is the m s e of an observation, that is, of M_1, M_2,   , then, since the m s e of $\frac{M_1}{a_1}$ is $\frac{\mu}{a_1}$, of $\frac{M_2}{a_2}$ is $\frac{\mu}{a_2}$,   the weights of these assumed observations are proportional to a_1^2, a_2^2,   .   Hence taking the weighted mean

$$X = \frac{\frac{M_1}{a_1}a_1^2 + \frac{M_2}{a_2}a_2^2 + \cdot + \frac{M_n}{a_n}a_n^2}{a_1^2 + a_2^2 + \quad + a_n^2}$$

$$= \frac{[aM]}{[a^2]}$$

Also, since $[a^2]$ is the weight of X,

$$\mu_X^2 = \frac{\mu^2}{[a^2]}$$

Cotes,[*] in solving this problem, reasons that, since for the same error of M the greater a is, the less is the error of X, we may take the coefficients a to express the relative weights of the values of X Now, placing the coefficients a_1, a_2 as weights along a straight line at distances $\frac{M_1}{a_1}$, $\frac{M_2}{a_2}$,   from one end of the line, the most probable

* *Harmonia Mensurarum*   Cambridge, 1722   (Quoted by Hultman, *Minsta Qvadr* Stockholm, 1860)

value will be the distance X of the centre of gravity of these weights from the end point, and will be found by taking moments about this point, that is,

$$X(a_1 + a_2 + \quad + a_n) = a_1\frac{M_1}{a_1} + a_2\frac{M_2}{a_2} + \quad + a_n\frac{M_n}{a_n}$$

and therefore

$$X = \frac{[M]}{[a]}$$

Ex To test the power of the telescope of the great theodolite (3 ft) of the English Ordnance Survey, and find the p e of an observation, a wooden framework was set up 12,462 ft distant from the theodolite when at station Ben More, Scotland It was so arranged that when projected against the sky a fine vertical line of light, the breadth of which was regulated by the sliding of a board, was shown to the observer The breadth of this opening was varied by half-inches from $1\frac{1}{2}$ in to 6 in during the observations which were as follows *

No of obs	Width	Side of opening	Mean of micr readings
1	6 0	{ left { right	{ 28 00 { 37 50
2	5 5	{ l { r	{ 28 50 { 37 00
3	5 0	{ l { r	{ 29 16 { 37 16
4	4 5	{ l { r	{ 30 16 { 36 66
5	4 0	{ l { l	{ 30 50 { 37 16
6	3 5	{ l { r	{ 31 16 { 37 00
7	3 0	{ l { r	{ 32 66 { 36 83
8	2 5	{ l { r	{ 33 50 { 36 83
9	2 0	{ l { r	{ 33 83 { 37 00
10	1 5	{ l { l	{ 35 50 { 37 16

Account of the Principal Triangulation, pp 54, 55

Let $X =$ the most probable value of the angle subtending an opening of 1 inch   Then we have the observation equations

$$6 X - 9\ 50 = v_1$$
$$5\ 5\ X - 8\ 50 = v_2$$
$$5\ X - 8\ 00 = v_3$$
$$4\ 5\ X - 6\ 50 = v_4$$
$$4\ X - 6\ 66 = v_5$$

$$3\ 5\ X - 5\ 84 = v_6$$
$$3\ X - 4\ 17 = v_7$$
$$2\ 5\ X - 3\ 33 = v_8$$
$$2\ X - 3\ 17 = v_9$$
$$1\ 5\ X - 1\ 66 = v_{10}$$

From the preceding we have for the individual values of X and their weights

$$X = 1\ 58 \quad \text{weight } 6^2$$
$$X = 1.55 \quad \text{weight } 5\ 5^2$$

$$\text{weighted mean} = \frac{1\ 58 \times 6^2 + 1\ 55 \times 5\ 5^2 +}{6^2 + 5\ 5^2 +}$$
$$= 1\ 55$$

or making the sum of the squares of the residuals v a minimum, that is,

$$(6X - 9\ 50)^2 + (5\ 5\ X - 8\ 50)^0 + \quad = \text{a min}$$

we find by differentiation that

$$X = 1\ 55$$

as before

The practical rule following from either method is the same, and may be stated thus   Multiply each observation equation by the coefficient of X in that equation, and add the products   The resulting equation gives the value of X

We again see that the principle of minimum squares is more general than that of the arithmetic mean, and why it was that we failed in solving the equations of Art 14   In other words, we cannot write in the preceding example

$$X = 1\ 58$$
$$X = 1\ 55$$

and take the mean of these values as the value of x, because these equations should be written strictly

$$X = 1\ 58 + \frac{v_1}{6}$$

$$X = 1\ 55 + \frac{v_2}{5\ 5}$$

where the errors are not lessened in the same proportion throughout the equations

Precision of a Linear Function of Independently Observed Quantities

64 Suppose that there are n independently observed quantities M_1, M_2,   whose m s e are μ_1, μ_2,   respectively, to find the m s e μ of F where

$$F = a_1 M_1 + a_2 M_2 + \quad + a_n M_n \qquad (1)$$

a_1, a_2, a_n being constants

This has been already solved in Art 19, where it was shown that

$$\mu^2 = [a^2 \mu^2]$$

On account of the great importance of this result we add another method of deriving it

If Δ_1, Δ_2,   . denote the errors of M_1, M_2,   . we shall have the true value T of F by writing $M_1 + \Delta_1$, $M_2 + \Delta_2$,   . for M_1, M_2,   in the above expression for F, that is,

$$T = a_1(M_1 + \Delta_1) + a_2(M_2 + \Delta_2) + \quad + a_n(M_n + \Delta_n)$$

Call Δ the error of F, then, since $T = F + \Delta$, we have

$$\Delta = a_1 \Delta_1 + a_2 \Delta_2 + \quad + a_n \Delta_n$$

and

$$\Delta^2 = a_1^2 \Delta_1^2 + a_2^2 \Delta_2^2 + \quad + 2 a_1 a_2 \Delta_1 \Delta_2 + \; .$$

Let the number of sets of M_1, M_2,   required to find T be n, and suppose Δ^2 summed for all the sets of values of Δ_1, Δ_2,   and the mean taken, then attending to Art. 20,

$$\mu^2 = a_1^2 \mu_1^2 + a_2^2 \mu_2^2 + \quad + 2 a_1 a_2 \frac{[\Delta_1 \Delta_2]}{n} + \qquad (2)$$

In forming all possible values of $\Delta_1 \Delta_2$, $\Delta_2 \Delta_3$, . . . , the number of values being very large, there will probably be as many $+$ as $-$ products of each form, and we therefore assume

$$[\Delta_1 \Delta_2] = [\Delta_2 \Delta_3] = . . \quad = 0$$

Hence

$$\mu^2 = [a^2 \mu^2] \qquad (3)$$

Ex 1 The Keweenaw Base was measured with two measuring tubes placed end to end in succession   Tube 1 was placed in position 967 times, and tube 2 966 times   Given the p e of the length of tube $1 = \pm 0^m \, 00034$, and of tube $2 = \pm 0^m \, 00037$, find the p e in the length of the line arising from the uncertainties in the length of the tubes

$$\begin{aligned}
[\text{ p e from tube } 1 &= 967 \times 0 \, 00034 = 0^m \, 329 \\
\text{p e from tube } 2 &= 966 \times 0 \, 00037 = 0^m \, 357 \\
\text{p e of line } &= \sqrt{0 \, 329^2 + 0 \, 357^2} \\
&= 0^m \, 485 \qquad]
\end{aligned}$$

Ex 2 In the Keweenaw Base the p e of one measurement of 94 tubes, deduced from the discrepancies of six measurements of these 94 tubes, was found to be $0^m \, 03$ Show that the p e in the length of the line of 1933 tubes arising from the same causes may be estimated at $\pm 0^m \, 136$

$$\begin{aligned}
[\text{ p e of 1 measurement of 1 tube} &= \frac{0 \, 03}{\sqrt{94}} \\[2mm]
\text{p e of base of 1933 tubes} &= \frac{0 \, 03}{\sqrt{94}} \sqrt{1933} \\[2mm]
&= \pm 0 \, 136 \quad]
\end{aligned}$$

Attention is called to these two problems, from the importance of the principles illustrated   In Ex 1 the p e of a tube was multiplied by the whole number of tubes to find the p. e. of the base from that cause, for the reason that with whatever error the tube is affected it is cumulative throughout the measurement.

In Ex 2 the p e of one tube is multiplied by the square root of the number of tubes, because each measurement is independent of every other, and the errors are as likely to be in excess as in defect, and, therefore, may be expected to destroy one another in the final result

Ex 3 The m s e of aM_1, μ being the m s e of M_1 and a a constant, is equal to $a\mu$, but the m s e of the sum of the a independently observed quantities M_1, M_2 M_a, that is, of $M_1 + M_2 + \quad + M_a$, the m s e of each being μ, is $\sqrt{a}\,\mu$ Explain

65 If the function F whose m s e is required is not in the linear form, we first reduce it to that form, as in Art 7, and apply Eq 3, Art. 64   Thus, if

$$F = f(M_1, M_2 \quad . \quad M_n)$$

the tiue value T of F will iesult if we write $M_1 + dM_1$, $M_2 + dM_2$, foi M_1, M_2, the differentials representing the eirois of these quantities   Then

$$T = f(M + dM,\ M_2 + dM_2,\quad\ )$$

Expanding by Taylor's theorem and retaining only the fiist powers of the small quantities dM_1, dM_2,   we have

$$T = F + \frac{\delta f}{\delta M_1} dM_1 + \frac{\delta f}{\delta M_2} dM_2 + \quad \cdot \quad + \frac{\delta f}{\delta M_n} dM_n$$

or

$$\text{Error of } F = a_1 dM_1 + a_2 dM_2 + \quad\ + a_n dM_n \quad (\text{1})$$

where $\qquad a_1 = \dfrac{\delta f}{\delta M_1},\ a_2 = \dfrac{\delta f}{\delta M_2},\qquad , a_n = \dfrac{\delta f}{\delta M_n}$

This expression is of the same form as (1), Ait 64   Hence

$$\mu_F{}^2 = a_1{}^2 \mu_1{}^2 + a_2{}^2 \mu_2{}^2 + \quad + a_n{}^2 \mu_n{}^2 = [a^2 \mu^2]$$

The still more general case of the m s e of a function of quantities which are themselves functions of the same observed quantities may be ieadily reduced to the form of Eq 1   The whole point is to express the eiror of F as a linear function of the eiiors of the independently observed quantities $M_1, M_2 \quad M_n$

It is useful to note that Eq 1 results from differentiating the function equation diiectly, as has been already pointed out in Art 7

Ex 1 If μ_1, μ_2 are the m s e of the measured sides *AB*, *BC* of a rectangle *ABCD*, find the m s e of the area of the rectangle [Here

$$F = M_1 M_2$$

by differentiation

$$dF = M_1 dM_2 + M_2 dM_1$$

and

$$\mu_F{}^2 = M_1{}^2 \mu_2{}^2 + M_2{}^2 \mu_1{}^2]$$

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Ex 2   The expansions of the steel and zinc bars of tube 1 of the Repsold base apparatus of the U S Lake Survey for 1° Fahr are approximately

$$S = 0\ 0248 \pm 0\ 0001 \overset{mm}{}$$
$$Z = 0\ 0617 \pm 0\ 0003 \overset{mm}{}$$

Show that

$$\frac{S}{Z} = \frac{2}{5} \pm \frac{1}{400} \text{ nearly}$$

[For

$$F = \frac{S}{Z}$$

$$dF = \frac{1}{Z} dS - \frac{S}{Z^2} dZ$$

and

$$(p\ e)^2 = \frac{1}{Z^2} (0\ 0001)^2 + \frac{S^2}{Z^4} (0\ 0003)^2]$$

Ex 3   The base b and the adjacent angles A, C of a triangle ABC are measured   If their m s e are respectively μ_b, μ_A, μ_C, find the m s e of the angle B and of the side a

To find μ_B
We have

$$B = 180 + \varepsilon - A - C$$

where ε denotes the spherical excess of the triangle

Hence, A and C being independent of one another,

$$\mu_B^2 = \mu_A^2 + \mu_C^2$$

To find μ_a

$$a = b \frac{\sin A}{\sin B}$$

By differentiation,

$$da = \frac{\sin A}{\sin B} db + b \frac{\sin (C - \varepsilon)}{\sin^2 B} \sin 1'' \ dA + a \cot B \sin 1'' \ dC$$

and therefore

$$\mu_a^2 = \frac{\sin^2 A}{\sin^2 B} \mu_b^2 + \frac{b^2 \sin^2 (C - \varepsilon)}{\sin^4 B} \sin^2 1'' \ \mu_A^2 + a^2 \cot^2 B \sin^2 1'' \ \mu_C^2$$

Ex 4   Given the base b and the angles A, B of a triangle with m s e μ_b, μ_A, μ_B respectively, to find the m s e μ_a of the side a
We have

$$a = b \frac{\sin A}{\sin B} \tag{1}$$

This might be expanded as in the preceding example, but more conveniently as follows

Take logarithms of both members   Then

$$\log a = \log b + \log \sin A - \log \sin B \tag{2}$$

(a) By differentiation,

$$da = \frac{a}{b} db + a \cot A \sin 1'' dA - a \cot B \sin 1'' dB$$

Hence

$$\mu_a{}^2 = \frac{a'}{b'} \mu_b{}' + a' \cot' A \sin' 1'' \mu_A{}^2 + a^2 \cot' B \sin' 1'' \mu_B{}^2 \qquad (3)$$

If, as is usually assumed in practice,

$$\mu_A = \mu_B = \mu, \text{ and } \mu_b = 0$$

then

$$\mu_a = a \sin 1'' \mu \sqrt{\overline{\cot' A} + \cot' B} \qquad (4)$$

(b) Using log differences as explained in Chapter I, we have by differentiating (2)

$$\delta_a \, da = \delta_b \, db + \delta_A \, dA - \delta_B \, dB \qquad (5)$$

where δ_a, δ_b are the differences corresponding to one unit for the numbers a and b in a table of logarithms, and δ_A, δ_b are the differences for $1''$ for the angles A and B in a table of log sines   Hence

$$\mu_a{}^2 = \left(\frac{\delta_b}{\delta_a}\right)^2 \mu_b{}^2 + \left(\frac{\delta_A}{\delta_a}\right)^2 \mu_A{}^0 + \left(\frac{\delta_b}{\delta_a}\right)^2 \mu_B{}^2 \qquad (6)$$

The two equations (3) and (6) may be used to check one another

Ex 5 The following example is given for the sake of showing the form of solution by the method of logarithmic differences

In the triangulation of Lake Superior there were measured in the triangle Middle, Crebassa, Traverse Id (*ABC*)

$$\angle A = 57° \ 04' \ 51'' 4 \quad \mu_A = 0'' \ 30$$
$$\angle B = 67° \ 15' \ 39'' 2 \quad \mu_b = 0'' \ 29$$

The side Middle-Traverse Id as computed from the Keweenaw Base is 16894 9 yards   Taking $\mu_b = 0 \ 05 \ yd$, find μ_a and μ_c

We have

$$a = b \frac{\sin A}{\sin B}$$

$$\log (a + da) = \log (b + db) + \log \sin (A + dA) - \log \sin (B + dB)$$

Then (see Ex 1, Art 7), the differences being expressed in units of the seventh decimal place,

$\log (b + db)$	$= 4 \ 2277556 + 257 \ db$
$\log \sin (A + dA)$	$= 9 \ 9239892 + 14 \ dA$
colog $\sin (B + dB)$	$= 0 \ 0351398 - \quad 9 \ dB$
$\log a + \delta_a \, da$	$= 4 \ 1868846 + 257 \ db + 14 \ dA - 9 \ dB$
and $283 \ da$	$= \quad 257 \ db + 14 \ dA - 9 \ dB$
Since $\log a$	$= 4 \ 1868846$, and 283 is the difference δ_a as given in the table

Hence $\qquad \mu_a{}^0 = \left(\dfrac{257}{283}\right)^2 (\;05)^0 + \left(\dfrac{14}{283}\right)^2 (\;30)^0 + \left(\dfrac{9}{283}\right)^2 (\;29)^0$

$\qquad\qquad\quad = 0\;0024$

and $\qquad \mu_a = \overset{yd}{0}\;05$

Also

$\qquad \mu_C = \sqrt{\mu_A{}^2 + \mu_B{}^2}$

$\qquad\qquad = \sqrt{(0\;29)^0 + (0\;30)^0}$

$\qquad\qquad = 0''\;42$

Miscellaneous Examples.

66 **Examples of Mean-Square and Probable Error.**

Ex 1 If in a theodolite read by 2 verniers the p e of a reading (mean of vernier readings) is 2″, show that if it is read by 3 verniers the p e of a reading will be a little over 1″ 5, and if read by 4 verniers a little less than 1″ 5

Ex 2 The p e of an angle of a triangle is r, show that the p e of the triangle error is $r\sqrt{3}$, all of the angles being equally well measured

$$[\text{Error} = 180^\circ - (A + B + C)]$$

Ex 3 The expansion of a bar for 1° C is $9a \pm 9r$, show that for 1° Fahr it is $5a \pm 5r$

Ex 4 The length of a measuring bar at the beginning of a measurement was $a \pm r_1$ After x measures had been made it was $b \pm r_2$ Show that the length of the nth measure, the length being supposed to change uniformly with the distance measured, is

$$a + \frac{n}{x}(b - a) \pm \sqrt{\left(1 - \frac{n}{x}\right)^2 r_1{}^0 + \frac{n^0}{x^2}r_2{}^2}$$

[For if da is the error of a, and db of b, then the error of $a + \dfrac{n}{x}(b - a)$ is $\left(1 - \dfrac{n}{x}\right)da + \dfrac{n}{x}db$, and the above p e follows

It is a common mistake to write the error in the form $da + \dfrac{n}{x}(db - da)$, and hence to infer that the p e is $\sqrt{r_1{}^0 + \dfrac{n^0}{x^2}(r_1{}^2 + r_2{}^2)}$]

Ex 5 Prove that the p e of the mean of two observations whose difference is d is 0 337 d, and the p e of each observation is 0 477 d

Ex 6 The line Monadnock-Gunstock (94469 *m*) was computed from the Massachusetts Base (17326 *m*) through the intervening triangulation   The p e of the line arising from the triangulation is ± 0^m 317, and the p e of the base is 0^m 0358 , find the total p e of the line

$$[\text{p e} = \sqrt{\left(\frac{94469}{17326} \times 0\ 0358\right)^2 + (0\ 317)^2} = \pm\ 0^m\ 372]$$

Ex 7 The Minnesota Point Base reduced to sea-level is

$$1325 \times 15 \text{ ft bar at } 32° + 11^m\ 314 \pm 0^m\ 421$$

and   15 ft bar at 32° = 179^m 95438 ± 0^m 00012

show that the p e of the base is ± 0^m 450

[p e = $\sqrt{(1325 \times 0\ 00012)^2 + (0\ 421)^2}$ = ± 0^m 450   We multiply ± 0^m 00012 by 1325 uncertain which sign it is , but whichever it is, it is constant all the way through]

Ex 8 If the zenith distance ζ of a star is observed n_1 times at upper culmination, and the zenith distance ζ' of the same star is observed n_2 times at lower culmination, show that the m s e of the latitude of the place of observation is

$$\frac{\mu}{2}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

μ being the m s e of a single observation

$$[\text{Latitude} = 90° - \tfrac{1}{2}(\zeta + \zeta')]$$

Ex 9 Given the telegraphic longitude results,

$$\begin{array}{ll}
 & \overset{h\quad m\quad s}{} \qquad \overset{s}{} \\
\text{Cambridge west of Greenwich} = 4\ 44\ 30\ 99 \pm 0\ 23 \\
\text{Omaha west of Cambridge} = 1\ 39\ 15\ 04 \pm 0\ 06 \\
\text{Springfield east of Omaha} = 25\ 08\ 69 \pm 0\ 11 \\
\hline
\text{Springfield west of Greenwich} = 5\ 58\ 37\ 34 \pm 0\ 26
\end{array}$$

$$[\text{p e} = \sqrt{23^2 + .06^2 + 11^2} = 0\ 26]$$

Ex 10 Given mass of earth + mass of moon = $\dfrac{1}{305879 \pm 2271}$

and   mass of moon = $\dfrac{1}{81\ 44}$ mass of earth

prove   mass of earth = $\dfrac{1}{309635 \pm 2299}$

$$[\text{For } (305879 \pm 2271) \times \frac{82\ 44}{81\ 44} = 309635 \pm 2299]$$

Ex 11   In measuring an angle suppose

$r_1 = $ p e of a pointing at a signal,

$r_2 = $ p e of a reading of the limb of the instrument,

$e = $ error of graduation of the arc read on,

then, assuming that these result from the only sources of error not eliminated, show if the limb has been changed m times and n readings taken in each position, that

$$\text{p e of angle} = \pm \sqrt{\frac{2(r_1^2 + r_2^2)}{mn} + \frac{e^2}{m}}$$

[For one position of the limb

$$\text{p e of angle} = \pm \sqrt{\frac{2(r_1^2 + r_2^2)}{n} + e}$$

as the error of graduation remains constant throughout each set of n readings]

Ex 12   The distance $o - 1^{mm}$ on a graduated line measure is read with a micrometer, show that the p e of the mean of two results is equal to the p e of a single reading

[For distance $o - 1^{mm}$

$= \frac{1}{2} \{(\text{first} + \text{second rdg}) \text{ at } o - (\text{first} + \text{second rdg}) \text{ at } 1^{mm}\}$

$(\text{p e})' = \frac{1}{4} \{4 (\text{p e})' \text{ of a rdg}\}$]

Ex 13   In the comparison of a mm space on two standards placed side by side and read with a micrometer, the p e of a single micrometer reading being a, show that the p e of the difference of the results of n combined measurements (each being the mean of two measurements) is $\sqrt{\dfrac{2}{n}} a$

[For   p e of a reading $= a$

p e of a combined measurement $= a$

and   p e of mean of n combined measurements $= \dfrac{a}{\sqrt{n}}$, etc]

Ex 14   A theodolite is furnished with n reading microscopes, all of the same precision   A graduation mark on the limb is read on m times with a single microscope, giving the p e of a single reading to be r_1 The telescope is then pointed at an object m times, and the p e of the mean of the microscope readings is found to be r_2 Show that the p e of a pointing is

$$\sqrt{r_2^2 - \frac{r_1^2}{n}}$$

[p e of reading (mean of verniers) with n microscopes $= \dfrac{r_1}{\sqrt{n}}$

Total error $=$ error of reading $+$ error of pointing

$$r_2^2 = \frac{r_1^2}{n} + (\text{p e of ptg})^2, \text{ etc]}$$

Ex 15 If $\mu_1 b_1$, $\mu_2 b_2$ are the m s e of the base measurements, and $\mu_3 \lambda$ the m s e of the ratio λ, given by the triangulation, of a base b_2 to a base b_1, show that the m s e of the discrepancy between the computed and measured values of b_2 is $b_2 \sqrt{[\mu^2]}$

$$
\begin{aligned}
&[\text{Discrepancy} = b_2 - b_1 \lambda = l \text{ suppose} \\
&\quad db_2 - b_1 d\lambda - \lambda db_1 = dl \\
&\text{and} \quad \mu_2{}^0 b_2{}^2 + b_1{}'(\mu_3 \lambda)' + \lambda'(\mu_1 b_1)^2 = \mu', \\
&\text{or} \quad b_2{}^0(\mu_2{}^0 + \mu_3{}^0 + \mu_1{}') = \mu'']
\end{aligned}
$$

Ex 16 At the time t_1 the correction to a chronometer was $a_1{}^s \pm \gamma_1$, and at the time t_2 it was $a_2{}^s \pm \gamma_2$, show that the p e of the rate of the chronometer is $\dfrac{\sqrt{\gamma_1{}^2 + \gamma_2{}^2}}{t_2 - t_1}$ and find the p e of the correction to the chronometer at an interpolated time t'

$$
\begin{aligned}
&\left[\text{Correction} = a_1 + \frac{a_2 - a_1}{t_2 - t_1}(t' - t_1) \text{ at time } t', \right. \\
&\quad \left. \text{p e} = \frac{\sqrt{(t_2 - t')^2 \gamma_1' + (t' - t_1)' \gamma_2'}}{(t_2 - t_1)} \right]
\end{aligned}
$$

Ex 17 Given
$$
\begin{aligned}
\lambda \cos \alpha &= l_1 \pm \gamma_1 \\
\lambda \sin \alpha &= l_2 \pm \gamma_2
\end{aligned}
$$
find p e of x and of α

$$
\begin{aligned}
&\left[dx = \frac{\delta \lambda}{\delta l_1} dl_1 + \frac{\delta x}{\delta l_2} dl_2 \right. \\
&\qquad = \frac{l_1}{\sqrt{l_1{}^2 + l_2{}^2}} dl_1 + \frac{l_2}{\sqrt{l_1{}^2 + l_2{}^2}} dl_2 \\
&\text{p e of } \lambda = \sqrt{\frac{l_1' \gamma_1{}^2 + l_2{}^2 \gamma_1'}{l_1' + l_2'}} \quad \text{Similarly p e of } \alpha = \left. \frac{\sqrt{l_2{}^2 \gamma_1' + l_1{}^2 \gamma_2{}^2}}{l_1{}^2 + l_2{}^2} \right]
\end{aligned}
$$

Ex 18 Given on a line measure the p e of a distance OA measured from O to be γ_1, and of OB, also measured from O, to be γ_2, find the p e of OD when D is the middle point of AB

$$
\begin{aligned}
&[\ OD = \tfrac{1}{2}(O A + OB) \\
&\quad r = \tfrac{1}{2} \sqrt{\gamma_1' + r_2{}^2}
\end{aligned}
$$

If $\gamma_1 = \gamma_2 = 1^\mu$, for example, then $r = 0^\mu 8$, when $\mu =$ one micron

It may at first sight appear paradoxical that the p e of the computed quantity may be smaller than the p e of the measured It is evident, however, that the error of OD is one-half the sum of the errors of OA and OB If the signs of the errors are alike the error of OD is *never* greater than the larger of the errors , if the signs are different it is *always* less]

Ex 19 Given the p e of x to be r, find the p e of log r

$$\left[\ d \log x = \frac{\text{mod}}{x}\ dx \right.$$

$$\left. \text{p e } \log x = \frac{\text{mod}}{x}\ r \right]$$

Ex 20 In the measurement of the Massachusetts base line, consisting of 2165 boxes, the p e of a box, as derived from comparisons with the standard meter, was $\pm$ 0^m 0000055, the p e from instability of microscopes in measuring a box was 0^m 000127, and the p e of the base from temperature corrections was 0^m 0332 Show that the p e of the base arising from these independent causes combined is 0^m 0358

$$[\text{p e } = \sqrt{(2165 \times 0^m\ 0000055)^2 + (0^m\ 000127\ \sqrt{2165})^2 + (0^m\ 0332)^2}$$
$$= \pm\ 0^m\ 0358]$$

Ex 21 Given the length of the Massachusetts base to be 17326^m 3763 $\pm$ 0^m 0358, show that the corresponding value of the p e of its logarithm is 8 973 in units of the seventh place of decimals

$$[\ \log (b \pm 0\ 0358) = \log b \pm \frac{\text{mod}}{b}\ (0\ 0358)\quad \text{See Art 7}$$

$$
\begin{array}{ll}
\log \text{mod} & 9\ 6377843 \\
\log 0\ 0358 & 8\ 5538830 \\
\hline
 & 8\ 1916673 \\
\log b & 4\ 2387077 \\
\hline
0\ 0000008973 & 3\ 9529596 \ ]
\end{array}
$$

Ex 22 The m s e of the log of a number N in units of the seventh decimal place is $\pm$ 10 6, find the ratio of the m s e to the number

$$[\ \log (N + v) = \log N + \frac{\text{mod}}{N}\ v$$

$$\mu^0_{\log (N + v)} = \left(\frac{\text{mod}}{N}\right)^2 \mu^0$$

$$\frac{\text{mod}}{N}\ \mu = 10\ 6 - 10^7$$

$$\text{and}\quad \frac{\mu}{N} = \frac{1}{410000}\ ]$$

67 **Examples of Weighting.**

Ex 1 The weights of the independently measured angles *BAC, CAD, DAE* are 3, 3, 1 respectively, find weight of the sum-angle *BAE*

$$\left[\frac{1}{\text{wt}} = \frac{1}{3} + \frac{1}{3} + \frac{1}{1},\quad \text{wt} = 0\ 6\right]$$

Ex 2 If $X = a_1x_1 + a_2x_2 + \quad + a_nx_n$, and $p_1, p_2, \quad p_n$ are the weights of $x_1, x_2, \quad x_n$, and P the weight of X, show that

$$\frac{1}{P} = \left[\frac{aa}{p}\right]$$

Ex 3 Prof Hall found, from observations of the satellites of Mars, that from Deimos, Mass of Mars $= \dfrac{1}{3095313 \pm 3485}$, and from Phobos, Mass of Mars $= \dfrac{1}{3078456 \pm 10104}$, the mass being expressed in the common unit Show that, taking the weighted mean, we have approximately

$$\text{Mass of Mars} = \frac{1}{3093500 \pm 3295}$$

Ex 4 On a graduated bar the space $0 - 1^m$ is measured and found to be $1m$ with a weight 1, and the space $0 - 2^m$ is measured and found to be $2m$ with a weight 2, required the value of the space $1^m - 2^m$ and its weight P

[Space $1^m - 2^m = 1m$ It makes no difference what the weights are so far as the value of the space is concerned

To find P $(1^m - 2^m) = (0 - 2^m) - (0 - 1^m)$

$$\frac{1}{P} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \text{ and } P = \frac{2}{3}]$$

Ex 5 Given the weight of $x = p$, show that

$$\text{weight of } \log x = \frac{x^2}{(\text{mod})^2} p$$

Ex 6 If $x = \dfrac{y}{c}$ and the weight of y is p, then

$$\text{weight of } 1 = c^2 p$$

Ex 7 Given the results for difference of longitude, Washington and Key West,

1873, Dec	24,		$19^m \ 01^s \ 42 \pm 0^s \ 044$
	Dec 26,		$1 \ 37 \pm \quad 037$
	Dec 30,		$1 \ 38 \pm \quad 036$
	Dec 31,		$1 \ 45 \pm \quad 036$
1874, Jan	9,		$1 \ 60 \pm \quad 046$
	Jan 10,		$1 \ 55 \pm \quad 045$
	Jan 11,		$19 \ 01 \ 57 \pm 0 \ 047$

show that

weighted mean $= 19^m \ 01^s \ 460 \pm 0^s \ 016$
weighted mean of first four nights $= 19 \ 01 \ 404 \pm 0 \ 019$
weighted mean of last three nights $= 19 \ 01 \ 573 \pm 0 \ 027$

and from the last two results check the first

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Ex 8 In the triangulation connecting the Kent Id Base, Md , and the Craney Id Base, Va , the length of the line of junction computed from

$$\text{Kent Id Base} = 26758\ 432 \overset{m}{\pm} \overset{m}{0}\ 38$$
$$\text{Craney Id Base} = 26758\ 176 \pm 0\ 43$$

Show that

(1) Discrepancy of computed values $= \overset{m}{0}\ 256 \pm \overset{m}{0}\ 57$
(2) Most prob length of junction line $= 26758\ 32\ \pm 0\ 028$

Ex 9 In latitude work with the zenith telescope, if n north stars are combined with s south stars, giving ns pairs, to find the weight of the combination, that of an ordinary pair, one north and one south, being unity
[Let $\mu =$ the m s e of an observation of one north star or of one south star

Then, as though combining the mean of n north stars with the mean of s south stars, the wt p of the combination is

$$\frac{1}{p} = \frac{\mu^2}{n} + \frac{\mu^0}{s}$$

But

$$1 = \frac{\mu^0}{1} + \frac{\mu^0}{1}$$

$$p = \frac{2ns}{n+s}\ ]$$

" The combination of more than two stars gave some trouble   In one case there were 3 north and 4 south, which would give 12 pairs, but with a weight of $2\dfrac{3 \times 4}{3+4}$ only   In this and all similar cases I treated the whole combination as one pair , that is, I inserted in the blank provided the half-sum of the mean of the declinations of north stars and of the mean of the declinations of south stars, and gave the result a higher weight   This is the only logical method " (Safford, *Report, Chief of Engineers U S A* , 1879, p 1987)
For a series of examples by Auy on the weights to be given to the separate results for terrestrial longitude determined by the observations of transits of the moon and fixed stars, see *Mem Roy Astron Soc* , vol xix

Ex 10 If a close zenith star is observed with a zenith telescope first as a north star, and immediately after as a south star, show that the weight of the resulting latitude is less than that found from observing an ordinary pair

Ex 11 In the triangulation of Lake Ontario the angle Walworth-Palmyra-Sodus was measured as follows
In 1875, with theodolite P and M No 1,
$74° 25' 05''\ 429 \pm 0''\ 29$, mean of 16 results ,
In 1877, with theodolite T and S No 3,
$74° 25' 04''\ 611 \pm 0''\ 22$, mean of 24 results ,
required the most probable value of the angle and its probable error

[With first theodolite p e of a single obs $= 0'' 29 \sqrt{16} = 1'' 16$

With second theodolite p e of a single obs $= 0'' 22 \sqrt{24} = 1'' 08$

Let a single result with the first theodolite be taken as unit of weight, then mean of 16 results has weight 16

Let a single result with the second theodolite have a weight p, referred to the same unit as the first, then mean of 24 results has weight $24 p$ The value of p is found from the relation

$$\frac{p}{1} = \left(\frac{1 \cdot 16}{1 \cdot 08}\right)^2$$

Also

most prob value of angle $= 74° 25' 04'' + \dfrac{1'' \cdot 429 \times 16 + 0'' \cdot 611 \times 24 p}{16 + 24 p}$

and

weight of this value $= 16 + 24 p$]

Note —If, instead of being two measurements of the same angle, the above were the measurements of two angles side by side, then

total angle $= 148° 50' 10'' \cdot 040$

because, no matter how much better one is measured than the other, we can do nothing but take the sum of the two values   The weight P of the result would be found from

$$\frac{1}{P} = \frac{1}{16} + \frac{1}{24 p}$$

Ex 12 An angle is measured n times with a repeating theodolite, and also n times with a non-repeating theodolite, the precision of a single reading and of a single pointing being the same in both cases, compare the weights of the results

[μ_1, μ_2 the m s e of a single pointing and of a single reading

With a non-repeating theodolite each measurement of the angle contains

(pointing + reading) — (pointing + reading)

(m s e)² of one measurement $= 2\mu_1^2 + 2\mu_2^2$

and (m s e)² of mean of n measurements $= \dfrac{1}{n} (2\mu_1^2 + 2\mu_2^2)$

With a repeating theodolite the successive measurements of the angle are

(pointing + reading) — pointing

pointing — pointing

pointing — (pointing + reading)

(m s e)² of n times the angle $= 2n\mu_1^2 + 2\mu_2^2$

and (m s e)² of the angle $= \dfrac{1}{n^2}(2n\mu_1^2 + 2\mu_2^2)$

If, then, p_1, p_2 denote the weights of an angle resulting from n reiterations or from n repetitions,

$$p_1 \quad p_2 = 1 + \frac{\mu_2^2}{n\mu_1^2} \quad 1 + \frac{\mu_2^2}{\mu_1^2}$$

and hence it would seem that the method of repetition is to be preferred to the method of reiteration This advantage is so much less, the smaller $\frac{\mu_2^2}{\mu_1^2}$ is , that is the more the precision of the circle leading increases in proportion to the precision of the pointing

This result is contradicted by experience—so much so that in all of the leading surveys repeating theodolites are no longer used in primary work Where, then, is the fault? Is the theory of least squares false? By no means We have only another example of a point which occurs over and over again, and which is so apt to be overlooked   (See Arts 53, 54)

The result obtained is true on the hypothesis that only accidental errors enter   We have assumed a perfect instrument   But the instrument maker cannot give what the geometer demands   From various mechanical reasons the systematic error in a repeating theodolite increases with the number of observations, whereas in the reiterating theodolite it disappears   This systematic error, in whatever way it arises, causes the trouble   It is useless to discuss accidental errors until it is out of the way, and as no means have yet been devised of getting rid of it, the instrument itself has been abandoned

Cf Struve, *Arc du Méridien*, vol 1 , Louis Cruls, *Discussion sur les Méthodes de répétition et reitération*, Gand, 1875 Herschel, *Outlines of Astronomy*, Art 188, *Coast Survey Report*, 1876, App 20]

Ex 13 An angle is measured with two repeating theodolites   With the first are made n_1 repetitions, μ_1, μ_1' being the m s e of a pointing and of a reading, with the second are made n_2 repetitions, μ_2, μ_2' being the m s e of a reading and of a pointing   Show that if n_1, n_2 are large the weights of the results are as

$$\frac{n_1}{\mu_1^2} \quad \frac{n_2}{\mu_2^2}$$

NOTE I

ON THE WEIGHTING OF OBSERVATIONS

68   When the sources of error are of such kinds that, so far as we know, they cannot be separated, the m s e and consequent weight are found as described in the preceding sections   The weight has been defined as a number representing the relative goodness of an observation, and as a

number inversely proportional to the square of the m s e
of an observation   If in a series of observations the con-
ditions required for the determination of the law of error
could be strictly fulfilled, these two statements would lead
to the same result   In actual cases, however, this is only
approximately true   Thus two separate determinations of
a millimeter space, made in the same way, gave

$$1000\ 1 \overset{\mu}{\pm} 0\ 40, \text{ mean of 20 readings}$$
$$1000\ 3 \pm 0\ 33, \text{ mean of 30 readings}$$

To find the weighted mean of these two sets of measure-
ments we may proceed in two ways   The number of re-
sults in the first measurement is 20, and the number in the
second is 30.   Hence, taking the weights proportional to
the number of results, the mean

$$= 1000 + \frac{20 \times 0\ 1 + 30 \times\ 3}{20 + 30} = 1000\ 220$$

Again, since the p e of the measurements are 0 40 and 0 33,
their weights are as $\dfrac{1}{40^2}$ to $\dfrac{1}{33^2}$, that is, as 1089 to 1600, and
the resulting weighted mean is 1000 219, agreeing, within
the limits of the p e , with the other value
    In this example the two methods agree as nearly as
could be expected from the small number of observations
But it is not always so   Some " run of luck," or balancing
of error, or constant conditions might have made the ob-
servations of one set fall very closely together, in which
case the weight as found from the p e would have been
very large, while varying conditions might have caused
wide ranges, giving a small weight   A great deal, there-
fore, depends on the judgment of the computer in deciding
what weight is to be given, it being constantly kept in mind
that the strict formulas which are correct in an ideal case
must not be pressed too far in practice   Thus in the second
set of observations above the first three results were 999 8,

999 8, 999 8   The p e computed from these would be zero,
and the consequent weight infinite   But no one will doubt
but that the mean of the 30 results is more reliable than the
mean of these three results

69 **An Approximate Method of Weighting.**—A
long-continued series of observations will show the kind of
work an instrument is capable of doing under favorable con-
ditions , and if work is done only when the conditions are
favorable, the p e derived from a certain number of results
will generally fall within limits that can be assigned *d priori.*
For example, with the Lake Survey primary theodolites,
which read to single seconds, the tenths being estimated,
the work of several seasons showed that the p e of the
mean of from 16 to 20 results of the value of a horizontal
angle, each result being the mean of a reading with telescope
direct and of a reading with telescope reverse, need not be
expected to be greater than 0″ 3   If, therefore, after having
measured a series of angles in a triangulation net with these
instruments, the p e all fell within ± 0″ 3, it was considered
sufficiently accurate to assign to each angle the same
weight

The objection to this is that "an instrument which has a
large periodic error may, if properly used, give as good
results as if it had none, but the discrepancies between its
combined results for an angle and their mean may be large,
thus giving an apparently large probable error to the mean.
Moreover, a given number of results over short lines, or
lines over which the distant signals are habitually steady
when seen in the telescope, will give a resulting value for
the angle of much greater weight than the same number of
combined results between two stations which are habitually
unsteady " *

The same method of weighting was employed by the
Northern Boundary Commission in their latitude work
" The standard number of observations [for a latitude de-
termination] was finally fixed at about 60, it being found

that with the 32-in instrument 60 observations would give a mean result of which the p e would be about 4 feet " *

70 **Weighting when Constant Error is Present.**— The preceding leads us to the case where the error of observation can be separated into two parts, one of which is due to accidental causes and the other to causes which are constant throughout the observations The total error e would, therefore, be of the form

$$e = l_1 + l_0$$

This case has been discussed already in general terms in Art 53 in explaining the well-known fact that an increase in the number of observations with a given instrument does not lead to a corresponding increase of accuracy in the result obtained

Let

$\mu_1 =$ the m s e of the observation arising from the accidental causes,

$\mu_2 =$ the error peculiar to the observation arising from the constant causes

Then μ_1, μ_2 being independent, the total m s e μ of observation may be assumed

$$\mu^2 = \mu_1^2 + \mu_2^2$$

If n observations have been made we shall have for the m s e μ_0 of their mean, since μ_2 is constant,

$$\mu_0^2 = \frac{\mu_1^2}{n} + \mu_2^2$$

It is evident that when n is large μ_2^2 becomes the important term, and that in any case the value of μ_0 and consequent weight can be but little improved by increasing the number of observations

For the purpose of finding the value of the m s e arising from the constant sources of error a special series of observations is in general necessary After this series has

been made the value of μ, found from it can be applied in the determination of the value of μ_o in any other series made under like conditions

For illustration let us consider a latitude determination with the zenith telescope   It is well known that with this instrument a latitude result found from two observations of a single star either north or south of the zenith is inferior to one found from a combination of a north and a south star. This arises, not from any difference in the mode of observation, but from the errors in declination as given in the star catalogue being cumulative in the one case and balancing in the long run in the other

The zenith-distance, ζ, of each star being observed, the half-difference of zenith-distances for each pair may be computed, and each of these computed values may be considered an observed value   The values of the declinations δ are taken from a catalogue of stars   The errors of δ are, therefore, independent of those of ζ, and are constant for the same pair of stars   The latitude φ from one pair is given by

$$\varphi = \tfrac{1}{2}(\delta' + \delta) + \tfrac{1}{2}(\zeta' - \zeta)$$

Let

$\mu_\zeta =$ the m s e of $\tfrac{1}{2}(\zeta' - \zeta)$ for one observation of one pair,
$\mu_\delta =$ the m s e of $\tfrac{1}{2}(\delta' + \delta)$ for this pair,
$\mu_\varphi =$ the m s e of the resulting latitude φ from one pair,

then for a single observation of this pair

$$\mu_\varphi^2 = \mu_\delta^2 + \mu_\zeta^2$$

and for n observations of this pair

$$\mu_\varphi^2 = \mu_\delta^2 + \frac{\mu_\zeta^2}{n}$$

The quantity μ_ζ will be found from repeated observations of the same pair of stars, as the error in declination will not influence the result   A better value will, of course, be obtained from several pairs than from a single pair. Let, then, many pairs of stars be observed night after night for

a considerable period   Collect into groups the latitudes resulting from the observed values of each separate pair Let n_1, n_2, n_m be the number of results in the several groups, the number in any group being at least two   Form the residuals for each group and compute the m s e in the usual way.   We have

No of Night	First Pair		Second Pair		
	Results	v	Results	v	
I	φ_1'	v_1'	φ_2'	v_2'	
2	φ_1''	v_1''	φ_2''	v_2''	
3	φ_1'''	v_1'''	φ_2'''	v_2'''	
Means	φ_1		φ_2		

Now, assuming that the m s e of observation of each pair is the same,

$$\mu_\varsigma^2 = \frac{[v_1 v_1]}{n_1 - 1}$$

$$\mu_\varsigma^2 = \frac{[v_2 v_2]}{n_2 - 1}$$

If, then, n is the total number of results, and m the number of groups, by adding the above equations there results

$$\mu_\varsigma^2 = \frac{[vv]}{n - m}$$

In finding μ_ϕ we assume that though errors of declination are constant for each star, still for a latitude found from many pairs in the same catalogue these errors may be

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regarded as accidental   Let, then, many different pairs of stars be observed on each of n nights at m' places, no star being observed at more than one place   Collect the means of the single results of each separate pair and form the residuals v' for each place, taking the differences between these means considered as single results and their mean for that place   Then, reasoning as above, the m. s e of a latitude resulting from n observations on a single pair of stars is

$$\mu_\phi = \sqrt{\frac{[v'v']}{n' - m'}}$$

where n' is the number of different pairs of stars observed and m' is the number of places occupied

Now, μ_δ is found from

$$\mu_\delta{}' = \mu_\phi{}^2 - \frac{\mu_\zeta{}^2}{n}$$

and is, therefore, known for the star catalogue used   This value may be taken in future work in finding μ_ϕ from

$$\mu_\phi{}' = \mu_\delta{}^2 + \frac{\mu_\zeta{}^2}{n}$$

and the consequent combining weight of φ will be as

$$\frac{1}{\mu_\phi{}^2}$$

An example of a similar kind is afforded in finding the weights of the angles measured with a theodolite in a triangulation where more rigid values are required than would be found by Art 69   The actual error of a measured value of an angle arises from two main sources, errors of graduation and errors of observation   The former are constant for different parts of the limb read on, and correspond to the declination errors above, while the latter are incapable of classification, and are, therefore, assumed to be accidental   The periodic errors of graduation are supposed to have

been eliminated by proper shiftings of the circle The resultant m s e μ of a single measurement is found from

$$\mu^2 = \mu_1^2 + \mu_2^2$$

and the m s e μ_o of the mean of n measurements made on the same part of the limb from

$$\mu_o^2 = \mu_1^2 + \frac{\mu_2^2}{n}$$

where μ_1, μ_2 are the m s e of graduation and observation respectively The method of treating this problem is quite similar to that of the preceding, μ_2 is found by reading the same graduation-mark on the limb many times, and μ_o by reading the angle between two fixed signals many times, the limb being changed after each reading Thence μ_1 is known for the instrument in question, and the combining weights of angles measured with this instrument are at once found

The foregoing leads to another important practical point in the measurement of angles If the weight of a single observation is unity, then the weight of the mean of n observations made with the limb in one position is

$$p = \frac{\mu_1^2 + \mu_2^2}{\mu_1^2 + \frac{\mu_2^2}{n}}$$

Experience has shown that we may safely assume

$$\mu_1 = \mu_2$$

and therefore it follows that

$$p = \frac{2n}{n+1}$$

Hence, no matter how many observations we make in one position of the limb, we never reach the precision of the mean of two observations made with the limb in different positions

It might fairly be inferred that the limb should be shifted after each single reading of an angle, and the rea-

sons for not doing so are to guard against mistakes in reading and to eliminate twist of station, as explained elsewhere (Art 54)

71 Assignment of Weight Arbitrarily.—So far we have deduced the combining weights from the observed values themselves, or from them in connection with a special series of observations   But this may not always be the best way of finding the weights   The observations may not be our only source of information, and, indeed, not the most reliable source   If, for example, some phenomenon has been observed by many persons in different parts of the country, and the observations are sent to one place for comparison and reduction, it would not be proper for the computer to deduce a weight for each series from the observations themselves independent of other sources of information he might have   Some of the most inexperienced observers with the poorest instruments might have apparently better results than the most experienced with good instruments. In such a case the computer must exercise his own judgment in classing the observations   He should consider the experience of the observer, his previous record for accurate work, the kind of instrument used, the conditions, and the observer's record of what he saw—whether it is clear and precise or hazy in its statements   An arbitrary scale of weights may then be constructed, and to each set of observations be assigned a weight from this scale according to the computer's estimate of its value   No two computers would be likely to assign precisely the same weights, but if done by one of experience and good judgment the result obtained from weighting in this way will undoubtedly be of more value than that found by the strict application of the formulas of least squares

The point is simply this   The class of observations considered may be expected to contain systematic errors which cannot be determined, and is therefore not capable of being treated by the method of least squares   As we have no direct means of eliminating this kind of error, we must do so

indirectly as best we can, and that is what the system of weighting mentioned seeks to accomplish

An example will be found in the discussion of the *Telescopic Observations of the Transit of Mercury*, May 5-6, 1878, Washington, 1879, where, of 109 observations sent in, to only 18 was the highest weight assigned   Prof Eastman, under whose direction they were reduced, says   "   Several instances may be found where small weight is given to observations that apparently agree well with those to which the highest weight is assigned, but in most cases the observer's remarks indicate the uncertain character of the observation "

72 **Combination of Good and Inferior Work.—** It is strictly in accordance with the idea of weight that if we have two results of very different degrees of accuracy, a result better on the whole than either may be found by combining both with their proper weights   But the proper weights may be difficult to find   On this account it depends on circumstances whether it is advisable to reduce a set of observations poorly made, in order to combine them with a well-made set   If the quantity is available for observing again it might not cost any more to do this than to reduce the poor observations   Even if it did the result would be more satisfactory   The committee of the Royal Society of England which was appointed to examine Col Lambton's geodetic work in India reported that " Col Lambton's surveys, though executed with the greatest care and ability, were carried on under serious difficulties, and at a time when instrumental appliances were far less complete than at present   There is no doubt that at the present time the surveys admit of being improved in every part   The standards of length are better ascertained than formerly, and all uncertainty on the unit of measure may be removed   The base-measuring apparatus can be improved. The instruments for horizontal angles used by Col Lambton were inferior to those now in use   The committee express the strong hope that the whole of Col.

Lambton's survey may be repeated with the best modern appliances " *

73 The Weight a Function of our Knowledge.— —If a quantity is not available for observing again, as, for example, some transient phenomenon, all of the material on hand must be used, and the best weights possible assigned to the separate values in order to combine them   The point is that where systematic or constant error has not been eliminated the weight to be assigned is a function of the state of our knowledge—is, in fact, a matter of individual judgment

This is brought out very fully in the methods used in combining the older star catalogues with the more modern ones   Thus Safford (*Catalogue of Mean Declinations of* 2018 *Stars,* Washington, 1879) says " In computing positions I have generally employed Argelander's rule giving to a modern determination from

    1 observation a weight $\frac{1}{2}$,
    2 observations a weight $\frac{3}{4}$,
    3 to 8 observations a weight 1,
    9 or more observations a weight $1\frac{1}{2}$ or 2

Argelander generally gives Piazzi a weight equal to unity, the value $\frac{1}{3}$ is much nearer the truth, in general he assigns rather a larger relative weight to the older and poorer observations than they deserve   But this is mostly compensated for by the number of determinations "

The weight of a quantity being a function of our knowledge may have assigned to it a certain value at one time and another value at another time when our knowledge of it has increased   Thus in the Fond du Lac (Wis) base, measured in 1872 with the Bache-Wurdemann compensating apparatus, a portion was measured seven times   The results differed widely, far beyond what was expected with the apparatus   No reason could be assigned at the time for the discordances   At this stage, then, one would have been

ment the weights may be taken inversely as the squares of these m s e

But if this has not been done, if in the older work instruments, observers, and methods were poorer than later and the two have to be combined in the adjustment, the computer must estimate as best he can their relative weights Thus in a system in Germany, France, and Austria reduced by Dr Albrecht ˙ the observations were made between the years 1863 and 1876 The methods of observation had been much improved in this interval In assigning the relative weights a scale of weights was first formed from a consideration of all the knowledge on hand, taking the march of improvement from year to year into account, and the separate determinations placed in one or other of these classes Thus, for example,

Weight 1— No change of observers, few observations, non-
adjustment of electric current,
Weight 2—No change of observers, usual variety of observa-
vations, non-adjustment of electric current,
Weight 3—Change of observers, usual variety of observa-
tions, non-adjustment of electric current,

and so on

Similarly Dr Bruhns in *Verhandlungen der europaischen Gradmessung*, 1880 See also *Coast Survey Report*, 1880, Appendix 6

74 **General Remarks.**—The subject of the weighting of observations is confessedly a difficult one In general it may be affirmed that the less experienced a computer is the more closely he will adhere to the rigorous formulas without considering whether systematic errors enter or not As he adds to his experience he will consider outside evidence as well as the evidence afforded by the observations themselves This will be specially true if he has any practical knowledge of how observations are made Indeed, it is doubtful if a computer can apply the principles of

* *Astronomische Nachrichten,* 2132

least squares properly unless he is at least an average observer

Great caution, however, is necessary in assigning weights, because it is sometimes possible so to choose them as to make observations tell anything desired   They should always be chosen from a consideration of all the evidence on hand, and may be changed as additional evidence is presented, so that a result is never final, but is ever open for improvement   The original records of the observations and the methods of reduction are quite as desirable, if indeed not more so, than the results deduced   The best plan, therefore, is to publish all of the data along with the reduction, when the reader, if he wishes, can make a reduction for himself   He can then form a more intelligent opinion of the computer's skill and judgment, and also of the value of the work

NOTE II

ON THE REJECTION OF OBSERVATIONS

75   There is nothing in the whole theory of errors more perplexing than the question of what shall be done with an observation of a series which differs widely from the others   In making a series of observations the observer is given full power   He can vary the arrangements, choose his own time for working, reject any result or set of results, he can do anything, in fact, that in his best judgment will tend to give the best value of the observed quantity   But when he has finished observing and goes to computing, has he the same power? Can he alter, reject, manipulate in such a way as in his best judgment will give a result of maximum probability ? As observer he was supreme, as computer is he supreme, or only in leading-strings ? Various answers

may be given to this question, as we look at it from one
point of view or another

In the hypothetical case on which the exponential law
of error was founded there were no discontinuous observa-
tions taken into account   There we contemplated not only
observations made with the best instruments and by the
most experienced observers, but observations of all grades,
from this highest grade down to those made with the poorest
instruments and by the most ignorant and careless observers
conceivable   It is only in this way that errors continuous
all the way from $+\infty$ to $-\infty$ could arise   In the cases
occurring in ordinary work we confine our attention to one
section of the observations only—that made with the good
instrument and by the skilful observer   This, to be sure, is
the most important, and, as shown in Art 30, the result
following from it differs ordinarily but little from that found
in the ideal case   But we are naturally confronted with
difficulty when we try to deal with a very incomplete series.
Extra assumptions must be made, and it is not to be won-
dered at that no solution yet offered is regarded as entirely
satisfactory

76 A common summary method of disposing of the sub-
ject is contained in the following statement   " The weights
[of the angles] would have been materially increased in
many instances by rejecting what would appear *bad* observa-
vations, but the rule has been never to reject any unless the
observer has made a remark to the effect that it ought to
be rejected " This statement, however, does not cover the
whole ground   Those who reason in this way make a dis-
tinction between mistakes and errors of observation.   Mis-
takes are rejected   But the great difficulty is to tell just
where mistakes end and errors begin

Given a set of measures involving discordances unlooked
for, and which the observer's remarks do not cover, how
shall we proceed? Two views may be taken   In the first
place, the computer, from a consideration of the measures
themselves and from all other evidence bearing on them

that he can discover, may make a distinction between measures and mistakes which will do for the set before him With observations of another kind he might have a different mode of procedure Another computer might have different rules altogether, precisely as in the case of weighting as explained in Arts 71-74

To put the discrepant values with the other values, and take the arithmetic mean of all, would give a result considerably different from what would be found by omitting them It would take a great many good observations to balance the effect of a single widely discrepant one It is, then, for the computer to judge whether this discrepant observation shall have the same weight as the others or a different weight

It may happen, indeed, that the discrepancy is so evidently a "natural mistake" that it may be corrected without a doubt from the evidence furnished by the other observations, and the discrepant observation changed so that it may be treated as a good one Thus an angle may be read 5' or 10' wrong, or a micrometer screw may be read 5 or 10 revolutions out of the way, as shown by the rest of the observations, and the like

Or the observations may be arranged in well-defined groups, and if the computer finds that he cannot account for the presence of unusual discrepancies in a certain group, he may decide to reject the whole group For example, in longitude work it may happen that one of the time stars may give a clock correction differing, say, one second of time from that given by fifteen or twenty others observed on the same night Instead of rejecting the single star it would perhaps be better to reject the whole night's work If necessary, an extra set of observations may be made to fill the blank By rejecting a group no hidden law can be slighted, for if any exists it will continue to reappear in further observations, and finally to reveal itself

77 In the second place, the computer, instead of trusting to his judgment, may call in the aid of the calculus of proba-

bilities and seek to establish a test or criterion for the rejec
tion of observations which will serve for all kinds of obser-
vations   Of the criterions which have been proposed the
earliest is due to Prof Peirce.   It is as follows   "Observa
tions should be rejected when the probability of the system
of errors obtained by retaining them is less than that of the
system of errors obtained by their rejection multiplied by
the probability of making so many and no more abnormal
observations"   A proof by Dr Gould will be found in the
U S Coast Survey Report, 1854, pp 131, 13'.   It is founded
on the assumption of the Gaussian law of error.

Another criterion " for the rejection of one doubtful ob
servation " is given by Chauvenet in his *Astronomy*, vol. ii
p 565   " We have seen that the function [Art 30]

$$\frac{2h}{\sqrt{\pi}} \int_0^{\frac{v}{?}} e^{?} \, d\Delta$$

represents in general the number of errors less than a which
may be expected to occur in any extended series of obser
vations when the whole number of observations is taken as
unity, r being the p e of an observation.   If this be multi
plied by the number of observations n, we shall have the
actual number of errors less than a; and hence the quantity

$$n - n\,\theta(t) \; - \; n\{1 - \theta(t)\}$$

expresses the number of errors to be expected greater than
the limit a But if this quantity is less than $\frac{1}{2}$ it will follow
that an error of the magnitude a will have a greater proba
bility against it than for it, and may, therefore, be rejected.
The limit of rejection of *a single doubtful observation* is, there
fore, obtained from the equation

$$\tfrac{1}{2} = n\{1 - \theta(t)\}$$

or

$$\theta(t) = \frac{2n - 1}{2n}\quad ,,$$

A third criterion was proposed by Mr. Stone, Radcliffe observer at Oxford, Eng , in *Month Not Roy Astron Soc* , 1868, 1873, in these terms " I assume that a particular person, with definite instrumental means and under given circumstances, is likely to make, on an average, one mistake in the making and registering *n* observations of a given class The probability, therefore, is that any record of his of this class of observations as a mistake is $\frac{1}{n}$ From the average discordances among the registered observations of this class we can find the p e of an observation in the usual way, and also the probability of an error greater than a given quantity, as C Then if the probability in favor of a discordance as large as C is less than that of a mistake, or $\frac{1}{n}$, the discordant observation is rejected "

The least objectionable criterion based on mathematical principles may, I think, be developed from the principle laid down in Art 50, where the maximum error was estimated at about five times the p e or three times the m s e If, therefore, an observation differs from the general run of the series by more than this amount it should at least be bracketed and attention be called to it

78 It may be stated that, as a general rule, criterions are apt to be most highly esteemed by those who look at the observations from a purely mathematical rather than from the practical observer's point of view The latter is, without doubt, the true standpoint Every observer will, consciously or unconsciously, construct a criterion suited to the sort of work he is engaged in This criterion will not necessarily be founded altogether on mathematical formulas Indeed, most likely it will not be Nor does it follow that the criterion adopted in any special series is of universal application or will receive universal assent In the process of weighting the observer will not assign weights always as the inverse square of the m s e It is often better to assign them arbitrarily from a feeling founded on a general grasp of all the circumstances connected with the making of the observations In like man-

ner, and on this same feeling, he will found his criterion for rejection   There is no uniform rule for weighting, neither is there one for rejection

A criterion such as Stone's, for example, would be very useful in the work for which it was proposed, and in its proposer's hands would be of much value   But that any one without the insight given by long familiarity into the special kind of work from which this criterion arose could apply it properly is not to be expected   As already pointed out, in the case of weighting the only thing for the computer to do is to publish all of the observations, including those rejected, along with his reduction, when, if more light can at any time be thrown on them, a new reduction can be made   Take, for example, Bradley's observations as reduced by Bessel and Auwers (Art 51)   It cannot be too strongly insisted on that a result deduced from a series of observations is never to be looked on as final, but as ever open for improvement

79 The difficulty in combining single observations lies in assigning to them their proper weights   We have assumed the arithmetic mean to give the most probable value If the sources of error could be separated, so that to each single observation could be ascribed its proper weight, the resulting weighted mean would be nearer the truth than the direct arithmetic mean   We can, therefore, *conceive* of a better value than the arithmetic mean in certain cases *

This has been already pointed out in Art 11

We may, therefore, consider whether, when discrepant observations occur we may not get a more satisfactory result by ignoring the arithmetic mean altogether   Suppose, for example, that we had three observed values, 100, 60, 61, and that we had no means of getting any further observations   These values would seem to show that the true value was likely to be nearer 60 than 100   Just

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* De Morgan (*Encyc Metrop* , " Theory of Probability," p 456) suggested that the combining weights might be found from the observed values themselves, but he did not develop his plan, and it is apparently fruitless

how much nearer is the question   To reject the observation 100 would be without reason, and to take the arithmetic mean of all three would ignore the evidence afforded by the observations themselves

The observations being discontinuous, the question is removed from the theory of least squares, which presupposes continuity (see Art 30), and must be treated by other methods   Laplace [x] discussed this problem long before Legendre and Gauss developed the method of least squares The subsequent confusion in the introduction of criterions has arisen from trying to fasten to the method of least squares what is in reality a very different question

When the observed values are discontinuous we cannot reasonably assume the value of the unknown sought to be a symmetrical function of them (see Art 11)   A plausible result has been given (Arts 11, 15) as that observed value which has as many observed values greater than it as it has less than it   Thus in the preceding example a good value to choose would be 61

In discussing such problems, so long as not much greater plausibility can be assigned to one method of combination than to another, the question of convenience of computation comes in   In respect to this the propriety of selecting the middle term stands pre eminent

80 It is ever to be kept in mind that unexpected discrepancies in his results do not always prove to the observer that his work is bad, any more than a close agreement among them shows it to be good   When either occurs great caution is necessary   It is unsafe to have a rigid rule of any kind for sifting the observed values, not allowing the computer to make use of evidence outside of the observations   By following such rules we are apt to bar the way to discovery of new truths, or at least to hinder progress in that direction   See, for example, the history of the discovery of personal equation, *American Cyclopædia*, vol xiii

* *Mém Acad Paris*, vol vi p 634

Throughout this discussion it has been assumed that the observations have been reduced by the observer himself or by a computer who is at the same time a competent observer A computer who is not an observer must of necessity employ the same criterion always, and in this case the criterion derived from Art 50, as stated on page 135, is to be preferred

The following memoirs may be consulted in addition to those already mentioned Gergonne, *Annales de Math*, vol xii pp 181 seq , Peirce in *Gould's Astron. Jour*, vol ii pp 161 seq , Airy in do, vol iv pp 137, 138, Winlock in do , vol iv pp 145–147, Stone, *Month Not Roy. Astron Soc*, vol xxviii pp 165 seq , vol xxxiv pp. 9 seq , Glaisher in do , vol xxxiii pp 391 seq , also in *Mem Roy Astron Soc*, vol xxxix pp 75 seq

CHAPTER IV.

Determination of the Most Probable Values

81   If direct measurements of a quantity have been made under the same circumstances, we have seen that the arithmetic mean of these measures gives the most probable value of the quantity   We now come to the case where the quantity measured is not the unknown required, but is a linear function of one or more unknowns whose values are to be found   This is the more general form, and its solution has been carried a certain distance in Art 14   The point stopped at was the combining of observations of different weights   As by the aid of the law of error this can now be done, we proceed to finish the solution

Let, as in Art. 14, the equations connecting a series of observed quantities M_1, M_2 . M_m, n in number, and the independent unknowns $X, Y,$., n_s in number ($n > n_s$), be

$$a_1X + b_1Y + . \quad - L_1 = M_1 + v_1$$
$$a_2X + b_2Y + \quad - L_2 = M_2 + v_2 \qquad (1)$$

$$a_nX + b_nY + \quad - L_n = M_n + v_n$$

where a_1 b_1, L_1, L_n are constants given by theory for each observation, and $v_1, v_2,$ v_n are the residual errors of observation

In practice the labor of handling these equations will be much lessened by using an artifice we have several times already employed (see Art 41)   Let $X', Y',$. be close approximations to the value of $X, Y,$ found by ordinary elimination from a sufficient number of the equations, or by some other method, as by trial, for example, and put

$$X - X' = x, \; Y - Y' = y,$$

where $x, y,$ are the corrections required to reduce the approximate values to the most probable values.

Then the observation equations reduce to

$$a_1 x + b_1 y + \quad - l_1 = v_1$$
$$a_2 x + b_2 y + \quad - l_2 = v_2 \qquad (2)$$

$$a_n x + b_n y + \quad - l_n = v_n$$

where

$$-l_1 = a_1 X' + b_1 Y' + \quad - L_1 \quad M_1$$
$$-l_2 = a_2 X' + b_2 Y' + \quad - L_2 - M_2$$
$$-l_n = a_n X' + b_n Y' + \quad . \quad - L_n - M_n$$

and are, therefore, known quantities

It is more convenient in practice to omit the residuals and write the observation equations in the form

$$a_1 x + b_1 y + \quad = l_1$$
$$a_2 x + b_2 y + \quad = l_2 \qquad (3)$$

$$a_n x + b_n y + \quad = l_n$$

always keeping in mind that the strict form is as in (2)

The observation equations 3 may be written

$$a_1 x = l_1'$$
$$a_2 x = l_2' \qquad (4)$$

$$a_n x = l_n'$$

if the values of $y, z,$ are supposed to be known.

Now, it has been shown in Art 63 that the most probable value of x would be found from these equations by taking the weighted mean of the separate values $\dfrac{l_1'}{a_1}, \dfrac{l_2'}{a_2}, \quad . \dfrac{l_n'}{a_n}.$ The weights of these values are in the same section shown to be as $a_1^2, a_2^2, \quad a_n^2$ We have, therefore,

$$x = \frac{[al']}{[aa]}$$

or, by putting for l_1', l_1', their values,

$$x = \frac{\lceil al \rceil - \lfloor ab \rfloor y - \lceil ac \rceil z -}{\lfloor aa \rfloor}$$

Similarly if the values of x, z, . are supposed to be known, the most probable value of y is found from

$$y = \frac{\lfloor bl \rfloor - \lfloor ba \rfloor x - \lfloor bc \rfloor z -}{\lfloor bb \rfloor}$$

and so on for z, .

Hence we should obtain the weighted mean values of x, y, . , that is, their most probable values, from the simultaneous solution of the equations

$$x = \frac{\lfloor al \rfloor - \lceil ab \rceil y - \lceil ac \rceil z -}{\lfloor aa \rfloor}$$

$$y = \frac{\lfloor bl \rfloor - \lfloor ba \rfloor x - \lceil bc \rceil z -.}{\lfloor bb \rfloor}$$

.

that is, from the simultaneous solution of the equations

$$\lceil aa \rceil x + \lfloor ab \rfloor y + \quad . = \lceil al \rceil$$
$$\lfloor ba \rfloor x + \lfloor bb \rfloor y + \quad . = \lfloor bl \rfloor \qquad (5)$$

. .

which equations are equal in number to the number of unknowns   They are called *normal equations*, or, better, *final equations*   We have thus found the most probable values of the unknowns in a series of observation equations by taking the mean *

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* Another method of solving a series of observation equations, due to Richelot, is worthy of notice

    Take the simple case of n equations involving two unknowns   Let the equations $(n > 2)$ be written in the strict form

$$a_1 x + b_1 y - l_1 = v_1$$
$$a_2 x + b_2 y - l_2 = v_2$$

$$a_n x + b_n y - l_n = v_n$$

to find the values of x, y which satisfy them best

    Multiply the equations in order by the undetermined factors k_1, k_2, k_n, and add, then if $k_1, k_2,$ k_n satisfy the condition $[bk] = 0$, we have

$$x = \frac{[kl]}{[ka]} + \frac{[kv]}{[ka]}$$

The best value of x must be that in which the second member is as small as possible, and this

The preceding result may be derived in a manner which will perhaps show still more clearly that we have taken the means of the separate values of the unknowns

For simplicity in writing consider the case of the three observation equations,

$$a_1 x + b_1 y = l_1$$
$$a_2 x + b_2 y = l_2$$
$$a_3 x + b_3 y = l_3$$

already given in Art 14

Solve in sets of two in all possible ways by the method of determinants, and

$$(a_1 b_2 - a_2 b_1)x = b_2 l_1 - b_1 l_2$$
$$(a_1 b_3 - a_3 b_1)x = b_3 l_1 - b_1 l_3$$
$$(a_2 b_3 - a_3 b_2)x = b_3 l_2 - b_2 l_3$$

Take the weighted mean of the values of x according to Art. 63, and the result is

$$x = \frac{(b_2 l_1 - b_1 l_2)(a_1 b_2 - a_2 b_1) + (b_3 l_1 - b_1 l_3)(a_1 b_3 - a_3 b_1) + (b_3 l_2 - b_2 l_3)(a_2 b_3 - a_3 b_2)}{(a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_2 b_3 - a_3 b_2)^2}$$

$$= \frac{(a_1 l_1 + a_2 l_2 + a_3 l_3)(b_1 b_1 + b_2 b_2 + b_3 b_3) - (a_1 b_1 + a_2 b_2 + a_3 b_3)(b_1 l_1 + b_2 l_2 + b_3 l_3)}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)(a_1 b_1 + a_2 b_2 + a_3 b_3)}$$

$$= \frac{[al][bb] - [ab][bl]}{[aa][bb] - [ab][ab]}$$

Similarly

$$y = \frac{[al][ab] - [bl][aa]}{[aa][bb] - [ab][ab]}$$

happens when $[la]$ is as great as possible   But $[la]$ as a linear function has no maximum unless a condition exists among the l's of at least two dimensions   The simplest such condition is

$$[l\lambda] = 1$$

If we now compute the maximum value of $[la]$ subject to the conditions

$$[b\lambda] = 0 \quad [\lambda\lambda] = 1$$

we shall find as the best value of x (Todhunter, *Diff Calc*, chap xvi)

$$x = \frac{[al][bb] - [bl][ab]}{[aa][bb] - [ab][ab]}$$

which agrees with the value resulting from the normal equations 5

These are the same values as would be derived from the solution of the equations

$$[aa]x + [ab]y = [al]$$
$$[ba]x + [bb]y = [bl]$$

Hence if in a series of n observation equations containing n_i independent unknowns $(n > n_i)$ we solve all possible sets of n_i equations by the method of determinants, the weighted means of the separate values found will be the most probable values of the unknowns

This method of solution would be very troublesome when the number of equations is large, and accordingly we must look for a more convenient but necessarily equivalent method

Since the principle that the sum of the squares of the residual errors is a minimum holds whether the observed quantity is a function of one or of several unknowns (Art 17), we can apply it to the simultaneous solution of the equations 2 The residual errors must satisfy the relation

$$v_1^2 + v_2^2 + \quad + v_n^2 = \text{a min}$$

that is, we must make

$$(a_1x + b_1y + \quad - l_1)^2 + (a_2x + b_2y + \quad - l_2)^2$$
$$+ \quad + (a_nx + b_ny + \quad - l_n)^2 = \text{a min}$$

Now, the variables $x, y,$ being independent, the differential coefficients of the expression for the minimum with respect to each of them in succession must be equal to zero. Hence

$$a_1(a_1x + b_1y + \quad - l_1) + a_2(a_2x + b_2y + \quad - l_2) + \quad = 0$$
$$b_1(a_1x + b_1y + \quad - l_1) + b_2(a_2x + b_2y + \quad - l_2) + \quad = 0 \quad (6)$$

or, collecting the coefficients of $x, y,$ in each equation,

$$[aa]x + [ab]y + [ac]z + \quad = [al]$$
$$[ab]x + [bb]y + [bc]z + \quad = [bl] \quad (7)$$
$$[ac]x + [bc]y + [cc]z + \quad = [cl]$$

The number of these equations is the same as the number of unknowns, that is, n_* Their solution will give the most probable values of x, y,   , and, adding these values to the approximate values X', Y' already known, the most probable values of X, Y,   will result

The equations 7 are identical in form with equations 5, found by taking the mean of all possible values of x and y As it has now been abundantly shown that the principle of the mean and that of minimum squares lead to the same results whatever be the number of independent unknowns, we shall use whichever promises to be most convenient in any particular case

It is useful to notice that equations 6 may be written

$$[av] = 0$$
$$[bv] = 0 \qquad (8)$$

These relations correspond to $[v] = 0$ in the case of the arithmetic mean, and may be used as a check on the computation of the values of the residuals

Ex Given the elevation of Ogden above the ocean by C P R R levels to be 4301 feet, and the elevation of Cheyenne to be 6075 feet, also the elevation of Cheyenne above Ogden by U P R R levels to be 1749 feet, find the adjusted elevations of Ogden and Cheyenne above the ocean, supposing the given results to be of equal value

First solution

    (a) Ogden—

            Direct determination   4301 weight 1

            Indirect determination 4326   "   ½

            weighted mean   = 4309 feet

    (b) Cheyenne—

            Direct determination   6075 weight 1

            Indirect determination 6050   "   ½

            weighted mean   = 6067 feet

Second solution

Let X, Y denote the elevations of Ogden and Cheyenne respectively Then

$$(X - 4301)^2 + (Y - 6075)^2 + (X - Y + 1749)^2 = \text{a min}$$

Differentiate with respect to X, Y in succession, and

$$2X - Y = 2552$$
$$- X + 2Y = 7824$$
$$X = 4309 \text{ feet}$$
$$Y = 6067 \text{ feet}$$

82 If the observation equations are of different weights $p_1, p_2, \quad . \quad p_n$, then, reducing each equation to the same unit of weight by multiplying it by the square root of its weight, we have

$$\sqrt{p_1}\, a_1 x + \sqrt{p_1}\, b_1 y + \qquad - \sqrt{p_1}\, l_1 = \sqrt{p_1}\, v_1$$
$$\sqrt{p_2}\, a_2 x + \sqrt{p_2}\, b_2 y + \qquad - \sqrt{p_2}\, l_2 = \sqrt{p_2}\, v_2 \qquad (1)$$

$$\sqrt{p_n}\, a_n x + \sqrt{p_n}\, b_n y + \qquad - \sqrt{p_n}\, l_n = \sqrt{p_n}\, v_n$$

with

$$[pvv] = \text{a min}$$

Substituting the values of $\sqrt{p_1}\, v_1, \sqrt{p_2}\, v_2, \quad .$ in the minimum equation, and differentiating with respect to x, y, as independent variables, we have the normal equations

$$[paa]x + [pab]y + \qquad = [pal]$$
$$[pab]x + [pbb]y + \qquad = [pbl] \qquad (2)$$

from which x, y, may be found

The relations for weighted equations corresponding to those of Eq 6, Art 81, are evidently

$$[pav] = 0, \; [pbv] = 0, \qquad\qquad (3)$$

Formation of the Normal Equations

83 Instead of forming the minimum equation and differentiating with respect to the unknowns in succession, it is more convenient to proceed according to the following plans suggested by the form of the normal equations themselves

The first, from equations 6, Art 81, may be stated as follows To form the normal equation in x multiply each observation equation by the coefficient of x in that equation, and add the results To form the normal equation in y multiply each observation equation by the coefficient of y in that equation, and add the results Similaily for the remaining unknowns

The second is suggested by the complete form of the normal equations as given in equations 7, Art 81 According to this plan we compute the quantities $\lfloor aa \rfloor, \lfloor ab \rfloor, \quad . \quad \lfloor al \rfloor,$ etc , separately, and wiite in their propei places in the equations

The equality of the coefficients of the noimal equations in the horizontal and vertical rows leads to a consideiable shortening of the numeiical work in computing these quantities Thus with three unknowns, x, y, z, all the unlike coefficients aie contained in

$$+ [aa]x + [ab]y + \lceil ac \rceil z = \lceil al \mid$$
$$+ [bb]y + \lfloor bc \rfloor z = \lfloor bl \mid$$
$$+ [cc \mid z = \lfloor cl \mid$$

Instead, therefore, of computing 12 quantities, only 9 aie necessary, as the remaining 3 can be at once wiitten down With n unknowns the saving of computation amounts to

$$1 + 2 + 3 + \quad + (n - 1) = \tfrac{1}{2} n(n - 1)$$

quantities

If the observation equations aie of diffeient weights the formation of the normal equations may be carried out precisely in the same way as in the pieceding as soon as the observation equations have been reduced to the same unit of weight

The foim of the weighted noimal equations, however, shows that it is not necessaiy, in oidei to obtain the coefficients $[paa]$, $[pab]$, to multiply the observation equations by the square ioots of their weights, and form the aux-

ılıaı y equations 1, Art 82, since the products aa, ab, multiplied by the weights of the respective equations from which they are foimed and summed, give $[paa]$, $[pab]$, directly   This is impoı tant because labor-saving

$Fı$ 1   Given the observation (or error) equations, all of equal weight,

$$x \qquad = 1$$
$$x + y \qquad = 3$$
$$x - y + r = 2$$
$$-x - y + z = 1$$

show that the normal equations are

$$4x + y \qquad = 5$$
$$x + 3y - 2z = 0$$
$$- 2y + 2z = 3$$

$Eı$ 2   The expansions x_1, x_1, r_1, x_4 for $1°$ Fahr of four standards of length were found by special experiment to be connected by the following relations at a temperature of $62°$ Fahı $(\mu = $ one micron $)$

				μ		
$+ x_1$				$= 39\ 945$	weight	1
	$+ x_2$			$= 5\ 932$	"	16
		$+ x_3$		$= 5\ 371$	"	4
		$+ x_2 - 1\ 0937 x_3$		$= 0\ 006$	"	8
	$+ 4 x_2$		$- x_4 = - 1\ 335$		"	3
$+ x_1$			$- x_4 = + 14\ 833$		"	6

find their most probable values

[The normal equations are

$$+ 7 r_1 \qquad\qquad - 6 x_4 = + 128\ 943$$
$$+ 72\ 000 x_2 - 8\ 750 x_3 - 12 x_1 = + 78\ 940$$
$$- 8\ 750 r_2 + 13\ 569 x_3 \qquad = + 21\ 432$$
$$- 6 x_1 - 12\ 000 x_2 \qquad + 9 x_4 = - 84\ 993$$

and $x_1 = 39\ 913,\ x_2 = 5\ 932,\ x_3 = 5\ 405,\ x_4 = 25\ 075]$

An example will now be given to illustrate the method of forming a series of observation equations

$Eı$ 3   At Washington the meridian transits of the following stars were observed to determine the correction and ıate of sidereal clock Kessels No 1324, April 12, 1870, at 11 hours clock time

20

Star	Observed clock time of transit, T	Right ascension of star, α
	h m s	s
τ Leonis	11 21 17 08	16 00
ν Leonis	11 30 20 41	18 51
β Leonis	11 42 28 57	20 57
o Virginis	11 58 38 15	36 20
η Virginis	12 13 18 37	16 37
θ Virginis	13 3 16 36	14 30

Let $x = $ corr of clock at 11 hours clock time,
$y = $ rate per hour of clock

Now, from theoretical considerations * it is known that the equation

$$x + y(T - 11) = \alpha - T$$

gives the relation between the clock correction and rate and the clock time of transit of each star observed

Hence the observation equations are

$$x + 0\ 35\,y = -1\ 96$$
$$x + 0\ 50\,y = -1\ 90$$
$$x + 0\ 71\,y = -2\ 00$$
$$x + 0\ 98\,y = -1\ 95$$
$$x + 1\ 22\,y = -2\ 00$$
$$x + 2\ 05\,y = -1\ 97$$
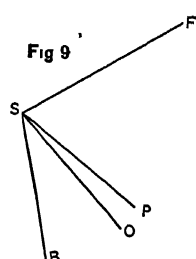
For the remainder of the solution see Art 104

Ex 4 In the triangulation of Lake Superior executed by the U S Engineers there were measured at station Sawteeth East the angles

Fig 9

Farquhar–Porcupine	62° 50′ 40″ 33	weight 5
Farquhar–Outer	64 11′ 44″ 02	" 7
Farquhar–Bayfield	100 20′ 26″ 12	" 4
Porcupine–Bayfield	37° 20′ 40 55	" 7
Outer–Bayfield	36 08′ 55 86	" 4

required the adjusted values of the angles

All of the angles may evidently be expressed in terms of FSP, FSO, FSB Let X, Y, Z denote the most probable values of these angles, and let X', Y', Z' be assumed approximate values of these most probable values, and x, y, z their most probable corrections Denoting the measured

angles in order by M_1, M_2, M_5, and their most probable corrections by v_1, v_2, v_5, we have

$$X' + x = X = M_1 + v_1$$
$$Y' + y = Y = M_2 + v_2$$
$$Z' + z = Z = M_3 + v_3$$
$$-X' - x + Z' + z = -X + Z = M_4 + v_4$$
$$-Y' - y + Z' + z = -Y + Z = M_5 + v_5$$

For simplicity the assumed approximate values may be taken equal to the observed values of the angles, so that we have the reduced observation equations

$$+ x = v_1 \quad \text{weight } 5$$
$$+ y = v_2 \quad \text{`` } 7$$
$$+ z = v_3 \quad \text{`` } 4$$
$$- x + z - 0.76 = v_4 \quad \text{`` } 7$$
$$- y + z - 1.66 = v_5 \quad \text{`` } 4$$

Hence the normal equations

$$12x - 7z = - 5.32$$
$$+ 11y - 4z = - 6.64$$
$$- 7x - 4y + 15z = + 11.96$$

Solving these equations, we find

$$x = - 0''.05, \quad y = - 0''.36 \quad z = + 0''.68$$

Hence $v_1 = - 0''.05$, $v_2 = - 0''.36$, $v_3 = + 0''.68$, $v_4 = - 0''.03$, $v_5 = - 0''.62$, and the adjusted values of the angles are

$$62° \ 59' \ 40''.28$$
$$64° \ 11' \ 34''.56$$
$$100° \ 20' \ 29''.80$$
$$37° \ 20' \ 49''.52$$
$$36° \ 08' \ 55''.24$$

We might have used v_1, v_2, v_5 for the corrections without introducing the symbols x, y, z at all

Ex 5 If the unknown x occurs in each of the n observation equations

$$- x + b_1 y + c_1 z + = l_1 \quad \text{weight } 1$$
$$- x + b_2 y + c_2 z + = l_2 \quad \text{`` } 1$$

these equations are equivalent to the reduced observation equations

$$b_1 y + c_1 z + = l_1 \quad \text{weight } 1$$
$$b_2 y + c_2 z + = l_2 \quad \text{`` } 1$$

$$[b]y + [c]z + = [l] \quad \text{`` } -\frac{1}{n}$$

[For the normal equations found from the first set after eliminating x are the same as the normal equations formed from the second set directly]

Ex 6 Instead of the observation equation

$$ax + by + cz + \quad = l \quad \text{weight } p$$

we may write

$$qax + qby + \quad = ql \quad \text{weight } \frac{p}{q^2}$$

84 **Control of the Formation of the Normal Equations.**—A very convenient check or control is the following   Add as an extra term to each observation equation the sum of the coefficients of x, y,   and of the absolute term l in that equation, and treat these added terms just as we do the absolute terms   Thus let s_1, s_2, s_n denote the sums, so that

$$a_1 + b_1 + c_1 + \quad + l_1 = s_1$$
$$a_2 + b_2 + c_2 + \quad + l_2 = s_2$$
$$.\qquad\qquad\qquad\qquad .$$
$$a_n + b_n + c_n + \quad + l_n = s_n$$

Multiply each of these expressions by its a and add the products, each by its b and add, and so on , then

$$[aa] + [ab] + \quad + [al] = [as]$$
$$[ab] + [bb] + \quad + [bl] = [bs]$$

$$[al] + [bl] + \quad + [ll] = [ls]$$

If these equations are satisfied the normal equations are correct   Thus each normal equation is tested as soon as it is formed

Since $[aa]$, $[ab]$, $[al]$ have been computed in forming the normal equations, the only new terms to be computed in applying the check are $[as]$, $[bs]$, $[ls]$, $[ll]$

Various modifications may readily be applied to suit individual tastes   Thus the absolute term may be placed

On the other side of the sign of equality, or the sign of the check may be changed so as to make the sum of each horizontal row equal to zero

85 Forms of Computing the Normal Equations.—
When the number of unknowns in the observation equations is large, or when their coefficients contain several figures, it is convenient to have a fixed form for the computation of the terms of the normal equations   It lightens the labor much either in forming, solving, or in finding the precision of the unknowns from these equations, if the computation is so arranged that a check can at all times be applied and the whole process proceed in a uniform and mechanical manner

The aids in the arithmetical work are a table of squares, a table of products, a table of reciprocals, a table of logarithms, and an arithmometer, or machine for performing multiplications and divisions   The latter is of the greatest use in computations of this kind   With it the *drudgery* of computation is in great measure got rid of   On the Lake Survey two forms of machine were used, the Grant and the Thomas   A series of trials showed that with either machine multiplications could be performed in from one-half to one-third of the time required with a log table, and with much less liability to error   About as good speed can be made for a short time with Crelle's multiplication-tables, but it cannot be kept up

Form (a) With Crelle's tables, or with a machine, the products aa, ab,   are found directly, and all that is then to be done is to write them in columns and take their sums $[aa]$, $[ab]$,   With a Thomas machine, however, each product may be added to all that precede, so that the final values result at once

Let us, for example, take the observation equations

$$-1\ 2x + 0\ 2y + 0\ 9 = v_1$$
$$+3\ 0x -_, 2\ 1y + 1\ 1 = v_2$$
$$+0\ 7x + 1\ 6y - 4\ 0 = v_3$$

Arrange as follows, the headings indicating the nature of the numbers underneath

a	b	l	s
$-1\ 2$	$+0\ 2$	$+0\ 0$	$-0\ 1$
$+3\ 0$	$-2\ 1$	$+1\ 1$	$+2\ 0$
$+0\ 7$	$+1\ 6$	$-4\ 0$	$-1\ 7$

aa	ab	al	as
$1\ 44$	$-0\ 24$	$-1\ 06$	$+0\ 12$
$9\ 00$	$-6\ 30$	$+3\ 30$	$+6\ 00$
$0\ 49$	$+1\ 12$	$-2\ 80$	$-1\ 19$
$+10\ 93$	$-5\ 42$	$-0\ 58$	$+4\ 93$

	bb	bl	bs
	$0\ 04$	$+0\ 18$	$-0\ 02$
	$4\ 41$	$-2\ 31$	$-4\ 20$
	$2\ 56$	$-6\ 40$	$2\ 72$
$-5\ 42$	$+7\ 01$	$-8\ 53$	$-6\ 94$

		ll	ls
		$0\ 81$	$-0\ 00$
		$1\ 21$	$+2\ 20$
		$16\ 00$	$+6\ 80$
$-0\ 58$	$-8\ 53$	$+18\ 02$	$+8\ 01$

Hence the normal equations, with the check all ready for solution, are

$$0 = + 10\ 93x - 5\ 42y - 0\ 58 \qquad +4\ 93$$
$$0 = -\ 5\ 42x + 7\ 01y - 8\ 53 \qquad -6\ 94$$

Form (b) If logarithms alone are used, form a table of the log coefficients of the observation equations as follows :

$$\log a_1, \log b_1, \qquad \log l_1, \log s_1$$
$$\log a_2, \log b_2, . \qquad \log l_2, \log s_2$$

$$\log a_n, \log b_n, \qquad \log l_n, \log s_n$$

Write $\log a_1$ on a slip of paper and carry it along the top row, forming the products

$$\log a_1 a_1, \log a_1 b_1, \qquad , \log a_1 l_1, \log a_1 s_1$$

Similarly with log a_2 foim the pioducts from the second
row,

$$\log a_2 a_2, \log a_2 b_2, \qquad , \log a_2 l_2, \log a_2 s_2$$

and so on till log a_n is reached

The numbers coiresponding to these logarithms are next
found, so that we have

$$a_1 a_1, a_1 b_1, \qquad , a_1 l_1, a_1 s_1$$
$$a_2 a_2, a_2 b_2, \qquad , a_2 l_2, a_2 s_2$$

$$a_n a_n, a_n b_n, \qquad , a_n l_n, a_n s_n$$

By addition we find

$$[aa], [ab], \qquad [al], [as]$$

the coefficients of the unknowns in the first noimal equation

Proceed in a precisely similar way with log b_1, log b_2,
log b_n, omitting the teim $[ab]$ already found, with
log c_1, log c_2, . log c_n, omitting the terms $[ac]$, $[bc]$ already
found, and so on till the last quantity is ieached,

log a	log b	log l	log s
0 07918n	9 30103	9 95424	9 00000n
0 47712	0 32222n	0 04139	0 30103
9 84510	0 20412	0 60206n	0 23045n
log aa	log ab	log al	log as
0 15836	9 38021n	0 03342n	9 07918
0 95424	0 79934n	0 51851	0 77815
9 69020	0 04922	0 44716n	0 07555n
	log bb	log bl	log bs
	8 60206	9 25527	8 30103n
	0 64444	0 36361n	0 62325n
	0 40824	0 80618n	0 43457n
		log ll	log ls
		9 90848	8 95424n
		0 08278	0 34242
		1 20412	0 83251

and the numbers coriesponding to these logs are exactly
the same as those in form (a) The remainder of the com-
putation is the same as there given

Form (c) If we wish to use a table of squares altogether, then since

$$ab = \tfrac{1}{2}\{(a+b)^{2} - a^{2} - b^{2}\}$$

and therefore

$$[ab] = \tfrac{1}{2}\{[(a+b)^{2}] - \lceil aa \rceil - \lceil bb \rceil\} \tag{1}$$

we form the square sums

$$[aa],\ [(a+b)^{2}],\ [(a+c)^{2}],\qquad [(a+l)^{2}],\ [(a+s)^{2}]$$
$$[bb],\ [(b+c)^{2}],\qquad\qquad [(b+l)^{2}],\ [(b+s)^{2}]$$
$$\vdots$$
$$[ll],\ [(l+s)^{2}]$$

and perform the necessary subtractions

In doing this, first take from the table of squares the squares *aa*, *bb*, . . *ll*, *ss*, and sum them, next write the co-efficients *a* of *x* on a slip of paper and carry them over the coefficients of *y*, *z*, , forming the sums $a_1 + b_1$, $a_1 + c_1$, $\cdots$, $a_2 + b_2$, $a_2 + c_2$, Take out the squares of these numbers and sum them Proceed similarly with the coefficients of *y*, *z*, . . Finish as indicated in (1)

Thus in the preceding example,

aa	bb	ll	ss
1 44	0 04	0 81	0 01
9 00	4 41	1 21	4 00
0 49	2 56	16 00	2 89
10 93	7 01	18 02	6 90

$a+b$	$(a+b)^{2}$	$a+l$	$(a+l)^{2}$	$a+s$	$(a+s)^{2}$
1 0	1 00	0 3	0 09	1 3	1 69
0 9	0 81	4 1	16 81	5 0	25 00
2 3	5 29	3 3	10 89	1 0	1 00
	7 10		27 79		27 69
$[aa]+[bb]=$	17 94	$[aa]+[ll]=$	28 95	$[aa]+[ss]=$	17 83
	−10 84		−1 16		9 86
	− 5 42		−0 58		4 03
	$=\ [ab]$		$=[al]$		$=[as]$

giving the same results as before

This form, which is very neat analytically, was first given by Bessel in the *Astron Nachr*, No 399

A consideration of the simple case of three observation equations, each involving two unknowns, will show that to form the normal equations, using a log table only, 24 entries in the table are required, while by this method we only need to enter a table of squares 18 times, thus effecting a saving of 6 entries   The Bessel method has also the advantage that, as we deal with squares, all thought with regard to sign is done away with   Besides, if the table of squares is a very extended one, accuracy can be had to a greater number of decimal places than with an ordinary log table   As compared with the logarithmic form, then, this method is to be preferred, more especially when the coefficients are not very different

On the other hand, if Crelle's tables or a computing machine is to be had, the direct process explained in (a) is much to be preferred to either, as experience will show

It is worth noticing that whichever method of formation of the normal equations is adopted, labor will be saved by changing the units in which the unknowns are expressed if the coefficients of the different unknowns are very different Thus, suppose we had the observation equations,

$$1000x + 0\ 0001y = 4\ 11 \qquad \begin{array}{c}\text{\textit{Check sums}}\\ 1004\ 1101\end{array}$$
$$999x + 0\ 0002y = 3\ 93 \qquad 1002\ 9302$$

from which to find x and y
By placing
$$x' = 100x,\ y' = 0\ 01y$$

the equations reduce to
$$\begin{array}{c}\\ \text{\textit{Check sums}}\end{array}$$
$$10x' + 0\ 01y' = 4\ 11 \qquad 14\ 12$$
$$9\ 99x' + 0\ 02y' = 3\ 93 \qquad 13\ 94$$

which are in more manageable shape for solution

21

Solution of the Normal Equations

Before beginning the solution of a series of normal equations we should consider whether the object is to find
(1) the unknowns only, or
(2) the unknowns and their weights,
and, in the latter case,
(a) whether the number of unknowns is large,
(b) whether many of the coefficients of the unknowns in the normal equations are wanting

Normal equations may be solved by the ordinary algebraic methods for the elimination of linear equations or by the method of determinants   When, however, they are numerous the methods of substitution and of indirect elimination, both introduced by Gauss, are more suitable.   Each has its advantages, which will be pointed out as we proceed   The method of substitution is quite mechanical, and is well suited for use with an arithmometer, which is as great a help in solving as it is in forming the normal equations.

86 **The Method of Substitution.** -For convenience in writing, take three unknowns, x, y, z, the process being the same whatever the number

The normal equations are

$$[aa]x + [ab]y + [ac]z = [al]$$
$$[ab]x + [bb]y + [bc]z = [bl] \qquad (1)$$
$$[ac]x + [bc]y + [cc]z = [cl]$$

From the first equation

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z + \frac{[al]}{[aa]} \qquad (2)$$

Substitute this value in the remaining equations, and, in the convenient notation of Gauss, there result

$$[bb\,1]y + [bc\,1]z = [bl\,1]$$
$$[bc\,1]y + [cc\,1]z = [cl\,1] \qquad (3)$$

where

$$[bb\ 1] = [bb] - \frac{[ab]}{[aa]}[ab]$$

$$[bc\ 1] = [bc] - \frac{[ab]}{[aa]}[ac]$$

$$[bl\ 1] = [bl] - \frac{[ab]}{[aa]}[al] \qquad (4)$$

$$[cc\ 1] = [cc] - \frac{[ac]}{[aa]}[ac]$$

$$[cl\ 1] = [cl] - \frac{[ac]}{[aa]}[al]$$

Again, from the first of equations 3,

$$y = -\frac{[bc\ 1]}{[bb\ 1]}z + \frac{[bl\ 1]}{[bb\ 1]} \qquad (5)$$

which value substituted in the second equation gives

$$z = \frac{[cl\ 2]}{[cc\ 2]} \qquad (6)$$

where

$$[cc\ 2] = [cc\ 1] - \frac{[bc\ 1]}{[bb\ 1]}[bc\ 1]$$

$$[cl\ 2] = [cl\ 1] - \frac{[bc\ 1]}{[bb\ 1]}[bl\ 1] \qquad (7)$$

Having thus found z, we have y at once by substituting in (5), and thence x by substituting for y and z their values in (2)

The first equations of the successive groups in the elimination collected are

$$[aa]x + [ab]y + [ac]z = [al]$$
$$[bb\ 1]y + [bc\ 1]z = [bl\ 1] \qquad (8)$$
$$[cc\ 2]z = [cl\ 2]$$

These are called the *derived normal equations.*

Divide each of these equations by the coefficient of its first unknown, and

$$x + \frac{[ab]}{[aa]}y + \frac{[ac]}{[aa]}z = \frac{[al]}{[aa]}$$

$$y + \frac{[bc\ 1]}{[bb\ 1]}z = \frac{[bl\ 1]}{[bb\ 1]} \qquad (9)$$

$$z = \frac{[cl\ 2]}{[cc\ 2]}$$

87 **Controls of the Solution.**—In solving a set of normal equations a control is essential   It is sometimes recommended to solve the equations arranged in the reverse order, when, if the work is correct, the same results will be found as before   But what is wanted in a control is a means of checking the work at each step, and not at the conclusion, it may be, of several weeks' work, when, if the results do not agree, all that is known is that there is a mistake somewhere without being able to locate it

(a) Continuation of the formation control   Experience has shown that it is convenient to carry on through the solution the check used in forming the equations   It merely consists in placing as an extra term to each equation the sums $[as]$, $[bs]$, $[ls]$, and operating on them in the same way as on the absolute terms $[al]$, $[bl]$,   The sum of the terms in every line, after each elimination of an unknown, must be each equal to the check sum numerically , the closeness of the agreement depending on the number of decimal places employed

This check may be applied at every step and mistakes be weeded out

(b) The diagonal coefficients $[aa]$, $[bb]$, .   of the normal equations, and $[aa]$, $[bb\ 1]$, $[cc\ 2]$,   of the derived normal equations, are always positive

For $\lfloor aa \rfloor$, $[bb\ 1]$, being the sums of squares, are positive Also

$$[aa][bb\ 1] = \begin{vmatrix} [aa], & [ab] \\ [ab], & [bb] \end{vmatrix} = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1, & b_1 \\ a_3, & b_3 \end{vmatrix}^2 +$$

a positive quantity

Similarly for $[cc\ 2]$, $[dd\ 3]$, . .

The principle may be of use as a check in the solution of a series of normal equations which are apparently correct, but which have been improperly formed   The following normal equations, which came up once in my own experience, will show this

$$7x + 7y - 9\ 1z = 26$$
$$7x + 28y - 12z = 64$$
$$-9\ 1x - 12y + 12z = -39$$

The derived normal equations are

$$7x + 7y - 9\ 1z = 26$$
$$+ 21y - 2\ 9z = 38$$
$$- 0\ 2z = 0\ 1$$

showing by the presence of the negative diagonal term that there was a mistake somewhere   An examination of the data from which the normal equations were derived showed a condition wanting and a condition incorrectly expressed. After the proper corrections had been made the solution was carried through without trouble

(c) By equations 8, Art 81, the residuals found by substituting for x, y, z their values in the observation equations must satisfy the relations

$$[av] = [bv] = \qquad = 0$$

(d) A very complete check is afforded by the different methods of computing $[vv]$, the sum of the squares of the residuals   (See Art 100)

88 **Forms of Solution.**—In applying the method of substitution to any special example it is important that the

arrangement of the computation be convenient and that every step be written down   Experience teaches that simplicity and uniformity of operation are great safeguards against mistakes

Form (a)   Solution without logarithms

The following form has been found by experience to be convenient   It is well fitted for use with the arithmometer or any other rapid method of multiplication   The form can be readily modified to suit computer's tastes

For illustration let us take, as before, three unknowns, x, y, z The computation is divided into sections, each section being formed in a precisely similar way, and in each section one unknown is eliminated

Given the normal equations,

No	x	y	z		Check	Remarks
I II III	$\begin{bmatrix} aa \\ ab \\ ac \end{bmatrix}$	$\begin{bmatrix} ab \\ bb \\ bc \end{bmatrix}$	$\begin{bmatrix} ac \\ bc \\ cc \end{bmatrix}$	$\begin{bmatrix} al \\ bl \\ cl \end{bmatrix}$	$\begin{bmatrix} as \\ bs \\ cs \end{bmatrix}$	

Solution

		y	z		Check	Remarks
IV	1	$\dfrac{[ab]}{[aa]}$	$\dfrac{[ac]}{[aa]}$	$\dfrac{[al]}{[aa]}$	$\dfrac{[as]}{[aa]}$	$\mathrm{I} \div [aa]$
V II		$[ab]\dfrac{[ab]}{[aa]}$ $[bb]$	$[ab]\dfrac{[ac]}{[aa]}$ $[bc]$	$[ab]\dfrac{[al]}{[aa]}$ $[bl]$	$[ab]\dfrac{[as]}{[aa]}$ $[bs]$	$\begin{matrix}\mathrm{IV} \quad [ab]\\ \mathrm{II}\end{matrix}$
VI		$[bb\ 1]$	$[bc\ 1]$	$[bl\ 1]$	$[bs\ 1]$	$\mathrm{II} - \mathrm{V}$
VII III		$[ac]\dfrac{[ab]}{[aa]}$ $[bc]$	$[ac]\dfrac{[ac]}{[aa]}$ $[cc]$	$[ac]\dfrac{[al]}{[aa]}$ $[cl]$	$[ac]\dfrac{[as]}{[aa]}$ $[cs]$	$\begin{matrix}\mathrm{IV} \times [ac]\\ \mathrm{III}\end{matrix}$
VIII		$[bc\ 1]$	$[cc\ 1]$	$[cl\ 1]$	$[cs\ 1]$	$\mathrm{III} - \mathrm{VII}$
IX	1	$\dfrac{[bc\ 1]}{[bb\ 1]}$	$\dfrac{[bl\ 1]}{[bb\ 1]}$	$\dfrac{[bs\ 1]}{[bb\ 1]}$	$\mathrm{VI} \div [bb\ 1]$
X VIII		$[bc\ 1]\dfrac{[bc\ 1]}{[bb\ 1]}$ $[cc\ 1]$	$[bc\ 1]\dfrac{[bl\ 1]}{[bb\ 1]}$ $[cl\ 1]$	$[bc\ 1]\dfrac{[bs\ 1]}{[bb\ 1]}$ $[cs\ 1]$	$\mathrm{IX} \times [bc\ 1]$
XI		$[cc\ 2]$	$[cl\ 2]$	$[cs\ 2]$	$\mathrm{VIII} \quad \mathrm{X}$
		1	$\dfrac{[cl\ 2]}{[cc\ 2]}$	$\dfrac{[cs\ 2]}{[cc\ 2]}$	$\mathrm{XI} \quad [cc]$

$$z = \frac{[cl\ 2]}{[cc\ 2]}$$

From Eq IX

$$y = -z\frac{[bc\ 1]}{[bb\ 1]} + \frac{[bl\ 1]}{[bb\ 1]}$$

From Eq IV

$$x = -y\frac{[ab]}{[aa]} - z\frac{[ac]}{[aa]} + \frac{[ab]}{[aa]}$$

To eliminate the first unknown, x In the first line write the quotients $\frac{[ab]}{[aa]}, \frac{[ac]}{[aa]},$ that is, the coefficients of the first normal equation divided by $[aa]$, the coefficient of x in that equation

The first line is now multiplied in order by $[ab]$, $[ac]$, forming the second and fifth lines

In the third and sixth lines write equations II and III

The fourth line is the sum of the second and third, and the seventh the sum of the fifth and sixth

This concludes the elimination of x, and the results in the fourth and seventh lines involve y and z only

Take now these results and proceed in a precisely similar way to eliminate y

The value of the last unknown, z, next results

Now proceed to find y and x Thus substitute for z its value in the eighth line, and we have y, and for y and z their values in the first line, and we have x

In carrying this solution into practice there are three points that deserve special notice

(1) In order to render the work mechanical, and so lighten the labor, the number of different operations should be made as small as possible   Instead, therefore, of dividing by $[aa]$, $[bb\ 1]$, $[cc\ 2]$, it is better to multiply by the reciprocals of these quantities, and, in order to avoid subtractions, to first change the signs of the reciprocals   We shall then have to perform only two simple operations—multiplication and addition   By transferring the terms $[al]$, $[bl]$, $[cl]$ to the left-hand side of the equations before beginning the solution, the values of the unknowns will come out with their proper signs

(2) Equations VI and VIII are the normal equations with x eliminated   An inspection of them shows that the coefficients of the unknowns follow the same law as the coefficients of the unknowns in the original normal equations with respect to symmetry of vertical and horizontal columns   Hence in the elimination it is unnecessary to compute these common terms more than once.   Thus $|bc\ 1|$ from Eq VI may be written down as the first term of Eq. VIII   This principle is of great use in shortening the work when the number of unknowns is large

(3) In a numerical example it is evident that since $|aa|$, $[bb\ 1]$, $[cc\ 2]$ do not in general divide exactly into the other coefficients of their respective equations, and that only approximate values of the unknowns can at best be obtained, it will give a closer result to divide by the larger coefficients and multiply by the smaller than *vice versa*   Attention to this by a proper arrangement of the coefficients before beginning the solution results in a considerable saving of labor, as the successive coefficients in the course of the elimination need not be carried to as many places of decimals to insure the same accuracy that a different arrangement would require

Ex   To make the preceding perfectly plain we shall solve in full the normal equations formed in Art 85

(1) Write the absolute term on the right of the sign of equality, and make the check sum equal to the sum of the other terms in each horizontal row

	x	y	l	Check	Remarks
I II	+ 10 93 − 5 42	− 5 42 + 7 01	+ 0 58 + 8 53	+ 6 09 + 10 12	
III	+ 1	− 0 496	+ 0 053	+ 0 557	I 10 93
IV II		+ 2 688 + 7 01	− 0 288 + 8 53	− 3 019 + 10 140	III II 5 42
V		+ 4 322	+ 8 818	+ 13 139	II − IV
VI		+ 1	+ 2 040 = y	+ 3 040	V + 4 322
VII III	+ 1	− 0 496 − 0 496	− 1 012 + 0 053	− 1 508 + 0 557	VI III 0 196
VIII	+ 1	0	+ 1 065 = x	+ 2 065	III VII

Hence

$$x = 1\ 06 \qquad\qquad y = 2\ 04$$

(2) Write the constant term on the left of the sign of equality, and form the check so as to make the sum of the terms in each horizontal line equal to zero

	Reciprocals	x	y	l	Check	Remarks
I II	0 0915	I 10 93 - 5 42	- 5 42 + 7 01	- 0 58 - 8 53	- 4 93 + 6 94	
III		- I	I 0 496	I 0 053	I 0 151	I × - 0 0915
IV II			- 2 688 I 7 010	- 0 289 - 8 530	- 2 445 + 6 910	III × - 5 42 II
V	0 2314		+ I 322	- 8 816	+ 4 195	II + IV
VI			- I	+ 2 040 = y	- I 040	V × - 02314
VII III		- I	- 0 496 + 0 496	+ I 012 I 0 053	- 0 516 + 0 451	VI × 0 496 III
VIII		- I		+ I 065 = x	- 0 065	III + VII

In order to find the values of the unknowns to two places of decimals the computation should be carried through to three places, and the third place dropped in the final result

Form (b) The logarithmic solution

As an example of the logarithmic method let us take the general form of the preceding example, when R and S are substituted for the absolute terms 0 58 and 8 53 respectively

In the numerical work it is better to convert all the divisions into multiplications Therefore write down the complementary logs of the divisors with the signs changed Each multiplier may now be written as needed on a slip of paper and carried over each logarithm to be operated on Thus for the first operation the slip would have on it 8 96138n, where the n indicates that the number is negative

Paper ruled into small squares, so as to bring the figures in the same vertical columns and facilitate additions and subtractions, renders the work more mechanical and is consequently an assistance to the computer.

In general solutions, when the number of unknowns is large, it will be found much better to carry a double check, one for the coefficients of $x, y,$ and the other for the coefficients of $R, S,$ Though unnecessary in our example, it is inserted for illustration

22

It will be noticed that the coefficients of R, S in the values of x, y follow the same law of symmetry as the normal equations   A little consideration will show that this is always the case

Hence, attending to this, we may shorten the computation by leaving out the common terms   We have, therefore, one term less to compute for each unknown, proceeding from the last to the first   The case is precisely analogous to that of Art 83

	x	y	Check	R	S	Check	Remarks
I	10 93	- 5 42	- 5 51	- 1		| I	
II	- 5 42	+ 7 01	- 1 59		- 1	| I	
III	1 09862	0 73400n	0 74115n	0		0	log I
IV	(8 96138n)	9 69538	9 70253	8 96138 +0 091		8 96138n -0 091	III - log 10 93 / Nos
V		0 42938n	0 43653n	9 69538n		9 695 8	IV - log 5 42
VI		- 2 668	- 2 732	- 0 496		1 0 496	Nos
II		+ 7 010	- 1 590		- 1	1 1	II
VII		+ 4 322	- 4 322	- 0 496	- 1	1 1 496	VI | II
VII₁		0 63568	0 63568n	0 69548n	0 0	0 17493	log VII
IX		(9 36432n)	0 00000	9 05980	9 36432	9 519257n	VIII - log 1 922
X		- 1	+ 1	+ 0 115	+ 0 231	- 0 346	Nos
XI				8 75518	9 05970	9 234639n	IX + log 54-
XII				+ 0 057	+ 0 115	- 0 172	Nos 10 01
XIII	- 1			+ 1 + 0 148	+ 0 115	- 0 091 - 0 263	

$$x = 0\ 148R + 0\ 115S$$
$$y = 0\ 115R + 0\ 231S$$

Substituting for R and S their values, we have, as before,

$$x = 1\ 06$$
$$y = 2\ 04$$

Ex 1   In the elimination of n normal equations by the method of substitution, show that the total number of independent coefficients in the original and derived normal equations is $\dfrac{n(n+1)(n+5)}{6}$

[The sum is $\frac{1}{2}\left\{1\ 4 + 2\ 5 + \quad n(n+3)\right\}$]

Ex 2   If the elimination of the unknowns in the normal equations is carried out by the method of substitution, the product

$$[aa]\,[bb\ 1]\,[cc\ 2]$$

has the same value whatever order has been followed

[For it is the determinant
$$
\begin{vmatrix}
[aa], & [ab], & [ac] \\
[ab], & [bb], & [bc] \\
[ac], & [bc], & [cc]
\end{vmatrix}
$$

89 **The Method of Indirect Elimination.**—This method is often of service when the number of unknowns is large and many of the terms in the normal equations are wanting   The principle involved in the solution is very simple   If x', y', z' are approximate values of x, y, z, and x_1, y_1, z_1 the corrections to these values, so that

$$
\begin{aligned}
x &= x' + x_1 \\
y &= y' + y_1 \\
z &= z' + z_1
\end{aligned}
$$

then, substituting these values in the normal equations, we have

$$
\begin{aligned}
[aa]x_1 + [ab]y_1 + [ac]z_1 &= [al]_1 \\
[ab]x_1 + [bb]y_1 + [bc]z_1 &= [bl]_1 \\
[ac]x_1 + [bc]y_1 + [cc]z_1 &= [cl]_1
\end{aligned}
$$

where the new absolute terms $[al]_1$, $[bl]_1$, $[cl]_1$ will be, on the whole, smaller than the original terms $[al]$, $[bl]$, $[cl]$ A second approximation will tend to decrease the absolute terms still farther.   The approximations are continued till the absolute terms either vanish or are sufficiently small   Then

$$
x = x' + x'' + \quad \cdot \quad , \; y = y' + y'' + \quad , \quad \cdot
$$

In finding the approximate values it is best to consider one unknown at a time, and preferably the equations should be operated upon in the order of magnitude of the absolute terms   Thus suppose $[al]$ the largest absolute term, then, since the diagonal term is in general the important term in a normal equation, we may neglect all but it in the equation and call $\dfrac{[al]}{[aa]}$ the first approximation to the value of x Substitute for x this value in the three equations, when a

value of y or of z may be similarly found Or the form of the equations may be such that first approximations to the values of both x and y, or of x, y, and , may be advantageously found from the original equations before making any substitutions These and similar points must be decided according to the circumstances of the case in hand

If the diagonal terms in the normal equations, instead of being much larger than the other terms, are but little larger than several of them, considerable difficulty will be found in making the approximations An approximation made for one unknown may counterbalance those made for several others, and the whole process will be found tedious and troublesome Various expedients have been suggested for getting over this difficulty, but in all cases where the normal equations are not very loosely connected (that is, where many terms are wanting) and the diagonal coefficients large, my experience has been that it is much better to use the method of substitution, or, in simple cases, the ordinary algebraic methods of elimination

While employed on the adjustment of the primary triangulation of Lake Superior we came across Von Freeden's glowing account * of the advantages of this method of solution, and determined to give it a trial. The number of equations to be solved was 32 Where the connection of the unknowns was loose it worked well enough, but where they fell in groups it was very troublesome and slow On the whole, the work was more fatiguing and took longer time We never tried the method again, though, as it works so nicely in simple cases, we were at one time very much in its favor

Gauss, the author of this form of solution, describes it as in general tedious (*langwierig*), and very justly For the contrary view see *Coast Survey Report*, 1855, App 4. Von

* *Die Praxis der Methode der kleinsten Quadrate* Braunschweig, 1863

Freeden, loc cit , Vogler, *Grundzuge der Ausgleichungsrech-nung*, p 129

Ex Required to solve the equations formed in Art 85

Solution

		Check	ι	*y*
10 93 — 5 42	— 5 42 + 7 01	— 5 51 — 1 59		
— 0 58 — 5 420 + 5 465	— 8 53 + 7 01 — 2 71	+ 9 11 — 1 59 — 2 755	0 5	1 0
— 0 535 + 5 465 — 5 420	— 4 230 — 2 71 + 7 01	+ 4 765 — 2 755 — 1 59	0 5	1 0
— 0 490 + 0 5465 — 0 1626	+ 0 070 — 0 271 + 0 2103	+ 0 420 — 0 2755 — 0 0477	0 05	0 03
— 0 1061 + 0 1093 — 0 0542	+ 0 0093 — 0 0542 + 0 0701	+ 0 0968 — 0 0551 — 0 0159	0 01	0 01
— 0 0510 + 0 0546	+ 0 0252 — 0 0271	+ 0 0258 — 0 0275	0 005	
+ 0 0036	— 0 0019	— 0 0017	1 005	2 04

The first two lines contain the coefficients of x, y, together with the check sums , the third line the absolute terms and their check sum

The first approximations are taken $y = 1$ and $x = 0 5$ The second and first lines are multiplied by these numbers and the products written in the fourth and fifth lines   The third, fourth, and fifth lines are summed, and $y = 1$, $x = 0 5$ are taken as the next approximations   The sum of the ι column of approximations gives the value of x, and of the y column the value of y

90 Combination of the Direct and Indirect Methods of Solution.—A numerical example illustrating an ingenious combination of the direct and indirect methods of elimination is given in the *Coast Survey Report* for 1878,

Appendix 8   For performing the multiplications necessary Crelle's tables are used altogether

In order to make the process employed readily followed I will give in general terms the solution of the three normal equations

$$[aa]x + [ab]y + [ac]z = [al]$$
$$[ab]x + [bb]y + [bc]z = [bl]$$
$$[ac]x + [bc]y + [cc]z = [cl]$$

according to this form

The coefficients and absolute term of the first equation are written in line 1, Table A   The reciprocal of the diagonal coefficient $[aa]$ is taken from a table of reciprocals and entered in the front column with the minus sign prefixed The remaining terms of line 1 are multiplied by this reciprocal, and the products written in line 2   This gives x as an explicit function of y and z

The coefficients and absolute term of Eq 2 (omitting the coefficient of x) are written in line 1, Table B   The terms in line 2, Table A, beginning with that under y, are multiplied by $[ab]$, the coefficient of y, and the products set down in line 2, Table B   The sum of lines 1, 2, Table B, is now written in line 3, Table A

Line 4, Table A, is found from line 3 in the same way as line 2 was found from line 1   This gives y as an explicit function of z

The coefficients and absolute term of Eq 3 (omitting the coefficients of x and y) are written in line 3, Table B   The terms in lines 2, 4, Table A, beginning with those under z, are multiplied by $[ac]$, $[bc 1]$, the coefficients of z in lines 1, 3 respectively, and the products set down in lines 4, 5, Table B   The sum of lines 3, 4, 5, Table B, is written in line 5, Table A

Line 6, Table A, gives the value of z

The next step is to find y and x The coefficients of the explicit functions are written in Table C   The absolute terms of the explicit functions are written in the first

line of Table D The value of z is multiplied by the coefficients of z in Table C, and the products written in the second line of Table D The sum of the numbers in column y gives the value of y written underneath in line 3 The value of y is multiplied by the coefficients of y in Table C, and the products written in the third line of Table D The sum of the numbers in column x gives the value of x

The values of x, y, z are now found to three places of decimals Denote them by x', y', z' If these values are not sufficiently close a second approximation must be made This we proceed to describe

First substitute the values obtained in the original normal equations, and carry out to a sufficient number of decimal places The residuals are written in the first line of Table E The coefficients in line 1, Table C, are multiplied by $- [al]_1$ and the products written in line 2, Table E The first reciprocal in Table A is multiplied by the same residual, and the product written in column x, line 1, Table F The sum of the numbers in column 2, Table E, is written underneath, as $- [bl\ 1]_1$

The coefficient in line 2, Table C, is multiplied by $- [bl\ 1]_1$ and the product written in line 3, Table E The second reciprocal in Table A is multiplied by the same residual, and the product written in column y, line 1, Table F The sum of the numbers in column 3, Table E, is written underneath, as $- [cl\ 2]_1$

The third reciprocal of Table A is multiplied by this residual, and the product written in column z, line 1, Table F This gives the correction to the value of z The first line of Table F corresponds to the first line of Table D, and exactly the same process employed in Table D will complete Table F The total values now are

$$x = x' + x''$$
$$y = y' + y''$$
$$z = z' + z''$$

A

Recip	x	y	z	
	$+[aa]$	$+[ab]$	$+[ac]$	$-[al]$
$-\dfrac{1}{[aa]}$	$x=$	$-\dfrac{[ab]}{[aa]}$	$-\dfrac{[ac_1]}{[aa]}$	$+\dfrac{[al]}{[aa]}$
		$[bb\,1]$	$+[bc\,1]$	$-[bl\,1]$
$-\dfrac{1}{[bb\,1]}$		$y=$	$-\dfrac{[bc\,1]}{[bb\,1]}$	$+\dfrac{[bl\,1]}{[bb\,1]}$
			$+[cc\,2]$	$-[cl\,2]$
$-\dfrac{1}{[cc\,2]}$			$z=+$	$+\dfrac{[cl\,2]}{[cc\,2]}$

B

y	z	
$+[bb]$	$+[bc]$	$-[bl]$
$-\dfrac{[ab]}{[aa]}[ab]$	$-\dfrac{[ac]}{[aa]}[ab]$	$+\dfrac{[ab]}{[aa]}[ab]$
	$[cc]$	$-[cl]$
	$-\dfrac{[ac]}{[aa]}[ac]$	$+\dfrac{[al]}{[aa]}[ac]$
	$-\dfrac{[bc\,1]}{[bb\,1]}[bc\,1]$	$+\dfrac{[bl\,1]}{[bb\,1]}[bc\,1]$

C

y	z
$-\dfrac{[ab]}{[aa]}$	$-\dfrac{[ac]}{[aa]}$
	$-\dfrac{[bc\,1]}{[bb\,1]}$

D

x	y	z
$+\dfrac{[al]}{[aa]}$	$+\dfrac{[bl\,1]}{[l\,b\,1]}$	$+\dfrac{[cl\,2]}{[cc\,3]}$
$-\dfrac{[ac]}{[aa]}\sim'$	$-\dfrac{[bc\,1]'}{[bb\,1]}\sim'$	$\sim'$
$-\dfrac{[ab]}{[aa]}y'$	y'	
x'	$\cdot$	

E

x	y	z
$-[al]_1$	$-[bl]_1$	$-[cl]_1$
	$+\dfrac{[ab]}{[aa]}[al]_1$	$+\dfrac{[ac]}{[aa]}[al]_1$
	$-[bl\,1]_1$	$+\dfrac{[bc\,1]}{[bb\,1]}[bl\,1]_1$
		$-[cl\,2]_1$

F

x	y	z
$+\dfrac{[al]_1}{[aa]}$	$+\dfrac{[bl\,1]_1}{[bb\,1]}$	$+\dfrac{[cl\,2]_1}{[cc\,2]}$
$-\dfrac{[ac]}{[aa]}z''$	$-\dfrac{[bc\,1]}{[bb\,1]}z''$	$\sim''$
$-\dfrac{[ab]}{[aa]}y''$	y''	
x''		

Addition of New Equations. It often happens that after
the adjustment of a long series of observations additional
observations are made leading to additional condition equa-
tions. To make a solution *de novo* is necessary, but the
work may be very materially shortened by the process just
given. Suppose, for simplicity, that one new condition has
been established. This will give one additional normal
equation, which may be written

$$[ad]x + \cdots + [sd]z + [dd]w = [dl] \qquad (1)$$

w being the new unknown.

The extra term to each of the other normal equations
may be written down at sight. The complete equations are

$$[aa]x + \cdots + [as]z + [ad]w = [al]$$
$$\cdots \quad + \quad [bs]z + [bd]w = [bl]$$
$$\quad + [sd]w = [sl]$$
$$+ [dd]w = [dl]$$

Now, values of $x, z \cdots$ have been already found from the
normal equations resulting from the original condition
equations, and these values may be taken as first approxi-
mations to the values of $x, z \cdots$ resulting from the above
four normal equations. Substitute in (1), and

$$[ad]x + \cdots + [sd]z + [dd]w = [dl] \qquad (2)$$

where $x, z \cdots$ are corrections to the approximate values of
$x, z \cdots$. The solution is now finished as follows:

Form Table C' by adding the extra column w to
Table C. The term $\dfrac{[dl]}{[aa]}$ is found by multiplying $[ad]$ by
the first reciprocal. The coefficients of the new equation,
C', are written in the first line of Table G. Since correc-
tions to values already found are required, the method of
proceeding must be similar to that employed in Table E.
The notation in Table C', G, and G explains this.

The reciprocal of the sum of column w, that is $[dd.1]$, in

Table G is written last in the column of reciprocals of Table C (a) with the minus sign   The product of this reciprocal and the absolute term $-[dl]'$ of the new equation, that is, $\frac{[dl]'}{[dd\ 3]}$, is an approximate value of w This value of w is multiplied by the terms in the last column of Table C (a), and the products are written in the first line of Table H   Column z gives the correction to z Table H is now completed in the same way as Tables D and F

C (a)

	y	z	w
$-\dfrac{1}{[aa]}$	$-\dfrac{[ab]}{[aa]}$	$-\dfrac{[ac]}{[aa]}$	$-\dfrac{[ad]}{[aa]}$
$-\dfrac{1}{[bb\ 1]}$		$-\dfrac{[bc\ 1]}{[bb\ 1]}$	$-\dfrac{[bd\ 1]}{[bb\ 1]}$
$-\dfrac{1}{[cc\ 2]}$			$-\dfrac{[cd\ 2]}{[cc\ 2]}$
$-\dfrac{1}{[dd\ 3]}$			

G

x	y	z	w
$[ad]$	$[bd]$	$[cd]$	$[dd]$
	$-\dfrac{[ab]}{[aa]}[ad]$	$-\dfrac{[ac]}{[aa]}[ad]$	$-\dfrac{[ad]}{[aa]}[ad]$
	$[bd\ 1]$	$-\dfrac{[bc\ 1]}{[bb\ 1]}[bd\ 1]$	$-\dfrac{[bd\ 1]}{[bb\ 1]}[bd\ 1]$
		$[cd\ 2]$	$-\dfrac{[cd\ 2]}{[cc\ 2]}[cd\ 2]$
			$[dd\ 3]$

H

x	y	z	w
$-\dfrac{[ad]}{[aa]}w$	$-\dfrac{[bd\ 1]}{[bb\ 1]}w$	$-\dfrac{[cd\ 2]}{[cc\ 2]}w$	$+\dfrac{[dl]'}{[dd\ 3]}$
$-\dfrac{[ac]}{[aa]}z'$	$-\dfrac{[bc\ 1]}{[bb\ 1]}z'$	z	w
$-\dfrac{[ab]}{[aa]}y'$	y		
x'			

91. Time required to Solve a Set of Equations.—

The labor involved in solving a series of normal equations, and the consequent time employed, increases enormously with an increase in the number of normal equations   To any one who has never been engaged in such work it will

seem out of all reason   Thus Dr Hugel,[x] of Hessen, Germany, states that he has solved 10 normal equations in from 10 to 12 hours, using a log table, but that 29 equations took him 7 weeks

The following are examples of rapid work   Gen Baeyer, in the *Kustenvermessung* (Vorwort, p vii) mentions that Herr Dase solved 86 normal equations between the first of June and the middle of September, and Mr Doolittle,[†] of the U S Coast Survey, solved 41 normal equations in 5½ days, or 36 working hours

A great deal depends, so far as speed is concerned, on the form of solution and on the mechanical aids used With a machine or with Crelle's tables much better time can be made than by the logarithmic method, which is by far the most roundabout.   Mr Doolittle used Crelle's tables and the form explained in the preceding article

In comparisons such as these it is important to know how many of the terms of the normal equations are wanting   Herr Dase's and Mr Doolittle's equations were both derived from triangulation work   The latter had only 430 terms in his 41 equations, while the former had 3141 different terms in his 86 equations

A machine for the solution of simultaneous linear equations has been invented by Sir W Thomson, of Glasgow, Scotland   The results are obtained, by a series of approximations, to any accuracy required   A description of the machine and of the mathematical principles underlying its construction will be found in Thomson and Tait's *Natural Philosophy*, vol 1 Appendix B   Sir W Thomson says " The exceeding ease of each application of the machine promises well for its real usefulness, whether for cases in which a single application suffices or for others in which the requisite accuracy is reached after two or three or more of successive approximations "

With regard to a machine for performing multiplica-

tions and divisions, I am certain that no one will ever prefer to use a log table in solving a set of equations by the method of substitution who has ever used any of the forms of arithmometer

92 For Jacobi's method of elimination, and the application of determinants to the solution of normal equations, see *Astron Nachr* , 404, 1960, *Month Not Roy Astron Soc* , vols xxxiv , xl , *Proc Amer Assoc for Adv of Science*, 1881

The Precision of the Most Probable (*Adjusted*) Values

93 The problem now before us is to find the m s e of the unknowns x , y ,   as determined from a series of normal equations   If the observation equations are reduced to the same unit of weight, which we shall take to be unity for convenience, the general form of the normal equations is

$$[aa]x + [ab]y + \qquad = [al \,|$$
$$[ab]x + [bb]y + \qquad = |bl\,] \qquad (1)$$

Let $\mu =$ the m s e of a single observation.
$\mu_x, \mu_y,$ $=$ the m s e of $x, y,$.
$p_x, p_y,$ $=$ the weights of $x, y,$

From Art 56 we have

$$p_x\mu_x^2 = p_y\mu_y^2 = \qquad = \mu^2 \qquad (2)$$

In order, therefore, to determine $\mu_x, \mu_y,$ we must make two computations, one of the weights $p_x, p_y,$ and the other of μ, the m s e of a single observation

It is evident from an inspection of the normal equations that $x, y,$ are linear functions of $l_1, l_1,$.   Let, then,

$$x = \alpha_1 l_1 + \alpha_2 l_2 + \,. \quad + \alpha_n l_n = [\alpha l \,|$$
$$y = \beta_1 l_1 + \beta_2 l_2 + \qquad + \beta_n l_n = [\beta l \,] \qquad (3)$$

in which $\alpha_1, \alpha_2,$, $\beta_1, \beta_1,$,   . are functions of $\alpha_1, b_1,$, $\alpha_2, b_2,$. , .   their values being as yet undetermined

Now, p being the m s c of each of the observed quantities $U, U_1, \ldots U_n$, must be also the m s c of $l_1, l_1, \ldots l_n$, which differ from $U, U, \ldots U_n$ by known amounts (see Art. 51). Hence since $l_1, l_1, \ldots l_n$ are independent of each other (Art. 101),

$$p' \quad p'' \quad \ldots, p_1 \quad p_1, \ldots $$ (4)

and therefore

$$p_1 \quad \frac{1}{p''}, p_1 \quad \frac{1}{p_1 p_1}$$ (5)

We shall first of all determine the weights p_1, p_1.

Before proceeding with the general proof let us consider the simple case of two unknowns, x, y, which are to be found from the three observation equations

$$a x + b y \quad l$$
$$a x + b y \quad l$$
$$a x + b y \quad l$$

all of the same weight.

The normal equations are

$$[aa] x + [ab] y \quad [al]$$
$$[ab] x + [bb] y \quad [bl]$$

Their solution by the method of substitution gives

$$x \quad \frac{[bl.1]}{[aa.1]}$$

$$\frac{1}{[bb.1]} \left(bl \quad \frac{[ab]}{[aa]} al \right)$$

$$\frac{1}{[bb.1]} \left(b \quad \frac{[ab]}{[aa]} a \right) l + \left(b \quad \frac{[ab]}{[aa]} a \right) l + \left(b \quad \frac{[ab]}{[aa]} a \right) l$$

Hence since l, l, l are independent of one another,

$$\frac{1}{p_y} \quad \frac{1}{[bb.1]} \left(b \quad \frac{[ab]}{[aa]} a \right) + \left(b \quad \frac{[ab]}{[aa]} a \right) + \left(b \quad \frac{[ab]}{[aa]} a \right)$$

$$\frac{1}{[bb.1]}$$ (6)

that is, *the weight of y is the denominator of the value of ;
found from the solution of the normal equations by the Gaussian
method of substitution.*

The general demonstration may be carried out more
simply by the application of the principles of undetermined
coefficients. Thus substitute $|al|, |al|$ for l_1, l_1,
in the normal equations (1), and

$$|aa||al| + |ab||al| + \quad . \quad |al|$$
$$|ab||al| + |bb||al| + \quad . \quad |bl|$$

or, arranging according to l_1, l_1.

$$\{[aa|\alpha_1 + [ab|\beta_1 \quad . \quad a_1\}l_1$$
$$\{[aa|\alpha_1 + [ab|\beta_1 \quad \quad$$

$$\{[ab|\alpha_1 + [bb|\beta_1 \quad b_1\}l_1$$
$$\{[ab|\alpha_1 + [bb|\beta_1$$

The unknown quantities α_1, α_n . .   may be so determined
that the coefficients of l_1, l_n . . shall each equal zero.
Hence the several sets of equations

$$\begin{cases} |aa|\alpha_1 + |ab|\beta_1 + \quad . . \quad a_1 \quad 0 \\ |aa|\alpha_1 + |ab|\beta_1 + \quad . . . \quad a_1 \quad 0 \\ \quad . \quad . \quad . \quad . \end{cases}$$

$$\begin{cases} |ab|\alpha_1 + |bb|\beta_1 + \quad . . . \quad b_1 \quad 0 \\ |ab|\alpha_1 + |bb|\beta_1 + \quad . \quad b \quad 0 \\ \quad . \quad . \quad . \quad . \end{cases}$$

are simultaneously satisfied by the same values of

$$\alpha_1 \, \alpha_n \quad . \quad . \, \beta_n \, \beta_n \ldots \,$$

Multiply the equations of each set by a_1, a_2,   in order, and add, then necessarily

$$[aa] = 1, \quad [a\beta] = 0, \quad [a\gamma] = 0, \qquad (9)$$

In a similar way, multiplying by b_1, b_2,   , c_1, c_2,   , etc, and adding, there result

$$[ba] = 0, \quad [b\beta] = 1, \quad [b\gamma] = 0,$$
$$[ca] = 0, \quad [c\beta] = 0, \quad [c\gamma] = 1,$$

Again, multiply the first set by a_1, a_2,   , the second by β_1, β_2,   , and so on, and add, and we have the sets of equations

$$\begin{cases} [aa][aa] + [ab][a\beta] + & = [aa] = 1 \\ [ab][aa] + [bb][a\beta] + & = [ba] = 0 \end{cases}$$
$$\begin{cases} [aa][a\beta] + [ab][\beta\beta] + & = [a\beta] = 0 \\ [ab][a\beta] + [bb][\beta\beta] + & = [b\beta] = 1 \end{cases} \qquad (10)$$

from which equations $[aa]$, $[a\beta]$,   may be found

It is plain that the coefficients of $[aa]$, $[a\beta]$,   , $[a\beta]$, $[\beta\beta]$, .   in these equations are the same as those of $x, y,$ in the normal equations, and that the absolute terms are $1, 0,$, $0, 1,$.   , instead of $[al]$, $[bl]$,   Hence,

94 To Find the Weights of the Unknowns.—*In the first normal equation write* 1 *for* $[al]$, *and in the other normal equations put* 0 *for each of* $[bl]$, $[cl]$,   , *the value of x found from these equations will be the reciprocal of the weight of x, and the values of y, z,   will be the values of* $[a\beta]$, $[a\gamma]$,   *In the second normal equation write* 1 *for* $[bl]$, *and in the other equations put* 0 *for each of* $[al]$, $[cl]$,.   , *the value of y found from these equations will be the reciprocal of the weight of y, and the values of z,   found will be the*

values of $|\beta\gamma|$, . Similarly for each of the unknowns in succession For example, the weight equations for three unknowns are

$[\alpha\alpha]$	$[\alpha\beta]$	$[\alpha\gamma]$			$[\alpha\beta]$	$[\beta\beta]$	$[\beta\gamma]$		$[\alpha\gamma]$	$[\beta\gamma]$	$[\gamma\gamma]$	
$[\alpha\alpha]$	$[\alpha b]$	$[\alpha c]$	1		$[\alpha\alpha]$	$[\alpha b]$	$[\alpha c]$	0	$[\alpha\alpha]$	$[\alpha b]$	$[\alpha c]$	0
$[\alpha b]$	$[bb]$	$[bc]$	0		$[\alpha b]$	$[bb]$	$[bc]$	1	$[\alpha b]$	$[bb]$	$[bc]$	0
$[\alpha c]$	$[bc]$	$[cc]$	0		$[\alpha c]$	$[bc]$	$[cc]$	0	$[\alpha c]$	$[bc]$	$[cc]$	1

The quantities $|\alpha\beta|$, $|\alpha\gamma|$, are necessary when the weight of a linear function of the unknowns is required, as will be seen presently (See Art. 101)

It is evident from the form of the weight equations that if the elimination is carried through by the method of substitution, the successive steps to the left of the sign of equality are the same as in Art 88 Hence if the equations are arranged so that the unknown whose weight is required , for example is found first, we should have the forms

	r				$[\alpha\gamma]$	$[\beta\gamma]$	rr	
$[\alpha\alpha]$	$[\alpha b]$	$[\alpha c]$	$[\alpha l]$		$[\alpha\alpha]$	$[\alpha l]$	$[\alpha l]$	n
	$[bb\,1]$	$[bc\,1]$	$[bl\,1]$		$[bb\,1]$	$[bl\,1]$	n
		$[cc\,2]$	$[cl\,2]$			$[cl\,2]$	1
		$[ll\,]$	$[rl\,3]$			$[ll\,3]$	1

Hence the coefficient of the unknown first found in the ordinary solution of the normal equations is the weight of that unknown By a separate elimination for each unknown the weight of that unknown could be found as above, but the process would be intolerably tedious.

95. *Special Cases of Two and Three Unknowns.* We may, however, from the preceding derive formulas for the weights

in a series of normal equations containing not more than three unknowns, which are easy of application.

Thus with two unknowns, and being found first,

In the reverse order, being found first,

or

where

With three unknowns, performing the elimination of the normal equations in the order we have

which expressions are easily transformed into

where

Fiom these foimulas the weights of the unknowns can be found directly without solving the normal equations   If the normal equations have simple coefficients it is much more rapid to find the weights in this way and solve the equations by ordinary algebra rather than by the Gaussian method   But when the number of unknowns exceeds three this becomes too cumbersome

Ex To find the weights of the adjusted angles in Ex 4, Art 83
Here

$$\lambda = 12 \times 11 \times 15 - 12 \times 16 - 11 \times 49$$
$$= 1249$$

and

$$p_x = \frac{1249}{149} = 8\ 4$$

$$p_y = \frac{1249}{131} = 9\ 5$$

$$p_z = \frac{1249}{132} = 9\ 5$$

If u_x, u_y, u_z denote the reciprocals of p_x, p_y, p_z respectively, then

$$u_x = 0\ 1193$$
$$u_y = 0\ 1049$$
$$u_z = 0\ 1057$$

96 *Modification of General Method* —To carry out the method of Ait 94 directly as stated would be excessively troublesome, and various modifications have been proposed The following scheme, which consists in running the weight equations together, will be found very convenient

Take, for simplicity in writing, three unknowns, x, y, z, and to the ordinary form of the normal equations as arranged for solution add the columns

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

the check being carried throughout

Perform the elimination exactly as stated in Art 88, and find the values of the unknowns in the usual way. We have then

x	y	z		R	S	T	Check
$\begin{bmatrix}aa\\ab\\ac\end{bmatrix}$	$\begin{bmatrix}ab\\bb\\bc\end{bmatrix}$	$\begin{bmatrix}ac\\bc\\cc\end{bmatrix}$	$\begin{bmatrix}al\\bl\\cl\end{bmatrix}$	1 0 0	0 1 0	0 0 1	$\begin{bmatrix}as\\bs\\cs\end{bmatrix}$ 1 1 1 1 1 1
1	$\begin{array}{c}\frac{[ab]}{[aa]}\\ [bb\,1]\\ [bc\,1]\end{array}$	$\begin{array}{c}\frac{[ac]}{[aa]}\\ [bc\,1]\\ [cc\,1]\end{array}$	$\begin{array}{c}\frac{[al]}{[aa]}\\ [bl\,1]\\ [cl\,1]\end{array}$	$\begin{array}{c}\frac{1}{[aa]}\\ R_1\\ \left\{\begin{array}{c}\frac{[ac]}{[aa]}\end{array}\right.\end{array}$	$\left\{\begin{array}{c}1\\0\end{array}\right.$	1	
	1	$\begin{array}{c}[bc\,1]\\ [bb\,1]\\ [cc\,2]\end{array}$	$\begin{array}{c}[bl\,1]\\ [bb\,1]\\ [cl\,2]\end{array}$	$\begin{array}{c}R_1\\ [bb\,1]\\ \left\{R_2\right.\end{array}$	$\begin{array}{c}1\\ [bb\,1]\\ \left\{S_2\right.\end{array}$	$\left\{\begin{array}{c}1\\1\end{array}\right.$	
		1	$\begin{array}{c}[cl\,2]\\ [cc\,2]\end{array}$	$\begin{array}{c}R_2\\ [cc\,2]\end{array}$	$\begin{array}{c}S_2\\ [cc\,2]\end{array}$	$\dfrac{1}{[cc\,2]}$	

where

$$0 = \frac{[ab]}{[aa]} + R_1$$

$$0 = \frac{[ac]}{[aa]} + \frac{[bc\ 1]}{[bb\ 1]} R_1 + R_2$$

$$0 = \frac{[bc\ 1]}{[bb\ 1]} + S_2$$

Now, taking the first column in the table under the heading R, and attending to Art 94, we have

$$[\alpha\gamma] = \frac{R_2}{[cc\ 2]}$$

$$[\alpha\beta] = \frac{R_1}{[bb\ 1]} - \frac{[bc\ 1]}{[bb\ 1]}[\alpha\gamma]$$

$$= \frac{R_1}{[bb.1]} + \frac{R_2 S_2}{[cc\ 2]}$$

$$u_x = [\alpha\alpha] = \frac{1}{[aa]} - \frac{[ab]}{[aa]}[\alpha\beta] - \frac{[ac]}{[aa]}[\alpha\gamma]$$

$$= \frac{1}{[aa]} + \frac{R_1^2}{[bb\ 1]} + \frac{R_2^2}{[cc\ 2]}$$

Similarly for the column under S,

$$|\beta\gamma| \qquad \begin{matrix} S \\ |a \cdot| \end{matrix}$$

$$u_y \quad |\beta\beta| \quad \begin{matrix} 1 \\ |bb.1| \end{matrix} \quad \begin{matrix} S' \\ |a \cdot| \end{matrix}$$

and for the column under T,

$$u_x - |\gamma\gamma| \quad \begin{matrix} 1 \\ |a.2| \end{matrix}$$

Also it is evident that

$$\begin{matrix} |d \cdot| \\ |a \cdot 2| \end{matrix}$$

$$y \quad \frac{|bl.1|}{|bb.1|} + \frac{|d \cdot|}{|a \cdot|} \cdot$$

$$x \quad \frac{|al|}{|aa|} + \frac{|bl.1|}{|bb.1|} k + \begin{matrix} \cdot \\ \cdot \end{matrix} k'$$

The forms of the expressions for $|aa|$, $|\beta\beta|$, $\cdots$ show that these quantities may be conveniently computed from the preceding tabular elimination scheme. Thus the sum of the products of each pair of numbers bracketed under the heads R, S, T will give u_x, u_y, u_z respectively.

The convenience of this form is seen in such a case as the following, which is of common occurrence. In a set of, say, 40 normal equations the weights of 10 of the unknowns may be required. These 10 would be placed last in the solution of the equations, and the extra columns k, k', $\cdots$ added after 30 of the unknowns had been eliminated, thus giving the weights required, with a trifling increase of work.

Ex 1 Given the normal equations

$$12x \qquad - 7z = R$$
$$\quad + 11y - 4z = S$$
$$- 7x - 4y + 15z = T$$

to find the weights of y and z

x	y	z	S	T
+ 12	0	− 7		
	+ 11	− 4		
− 7	− 4	+ 15		
1	0	− 0 5833		
	+ 11	−4	+ 1	
	− 4	+ 10 9169		} 1 1
	1	− 0 3636	+ 0 0909 }	
		+ 9 4625	+ 0 3636 }	⊢ 1 }
		1	+ 0 0384 }	+ 0 1057 }

Hence

$$u_z = 1 \times 0\ 1057 \qquad\qquad = 0\ 1057$$
$$u_y = 1 \times 0\ 0909 + 0\ 3636 \times 0\ 0384 = 0\ 1049$$

agreeing with the values in Art 95

Ex 2 Show that

$$[bl\ 1] = [al]R_1 + [bl]$$
$$[cl\ 2] = [al]R_2 + [bl]S_2 + [cl]$$

. . .

Ex 3 Show that the multipliers R_1, R_2, . . satisfy the conditions

$$[aa]R_1 + [ab] = 0$$

$$[aa]R_2 + [ab]S_2 + [ac] = 0$$
$$[ab]R_2 + [bb]S_2 + [bc] = 0$$

$$[aa]R_3 + [ab]S_3 + [ac]T_3 + [ad] = 0$$
$$[ab]R_3 + [bb]S_3 + [bc]T_3 + [bd] = 0$$
$$[ac]R_3 + [bc]S_3 + [cc]T_3 + [cd] = 0$$

.

97 Second Method of Finding the Weights of the Unknowns.—If we multiply the first of the normal equations 1, Art 93, by $[\alpha\alpha]$, the second by $|\alpha\beta|$, the third by $[\alpha\gamma]$, and so on, add the products, and attend to equations 10, Art 93, we obtain

$$x = [\alpha\alpha][al] + [\alpha\beta][bl] + |\alpha\gamma||cl| +\cdot$$

Similarly

$$y = [\alpha\beta][al] + [\beta\beta][bl] + |\beta\gamma||cl| +\quad .$$
$$z = [\alpha\gamma][al] + [\beta\gamma][bl] + |\gamma\gamma||cl| +\cdot\qquad .$$

Hence since $[\alpha\alpha]$, $[\beta\beta]$,   are the reciprocals of the weights of $x, y,$, this method of finding the weights may be stated as follows

In any given series of observations, having formed the normal equations, replace the numerical absolute terms by the general symbols $[al]$, $[bl]$, *. and find by any method of elimination the values of* $x, y,$ *, in terms of* $|al|$, $|bl|$, *, then the weight of x is the reciprocal of the coefficient of* $|al|$ *in the value of x, the weight of y is the reciprocal of the coefficient of* $|bl|$ *in the value of y, and so on*

The coefficients of the remaining symbols for the absolute terms in the expressions for $x, y,$ give the values of $[\alpha\beta]$, $[\alpha\gamma]$,   , $[\beta\gamma]$,   , and the numerical values of the unknowns $x, y,$ may be found by substituting for $[al]$, $[bl]$,   their numerical values

In this method of computing the values of the unknowns and their weights a machine can be used with great advantage

The formulas of Art 96 are easily derived from the preceding principles   For solving the normal equations

$$[aa]x + [ab]y + [ac]z = |al|$$
$$[ab]x + [bb]y + [bc]z = |bl|$$
$$[ac]x + [bc]y + [cc]z = |cl|$$

by the method of substitution we have for the first unknown

$$x = \frac{[al]}{[aa]} + \frac{[bl\ 1]}{[bb\ 1]} R_1 + \frac{[cl\ 2]}{[cc\ 2]} R_2$$

$$= [al]\left(\frac{1}{[aa]} + \frac{R_1^2}{[bb\ 1]} + \frac{R_2^2}{[cc\ 2]}\right)$$

$$+ [bl]\left(\frac{R_1}{[bb\ 1]} + \frac{R_2 S_2}{[cc\ 2]}\right) + [cl]\frac{R_2}{[cc\ 2]}$$

Comparing this with the general expression for x in Eq 1,

$$[aa] = \frac{1}{[aa]} + \frac{R_1^2}{[bb\ 1]} + \frac{R_2^2}{[cc\ 2]}$$

$$[a\beta] = \frac{R_1}{[bb\ 1]} + \frac{R_2 S_2}{[cc\ 2]}$$

$$[a\gamma] = \frac{R_2}{[cc\ 2]}$$

Similarly for y and z

Ex To find the weights of the unknowns in Ex 4, Art 83
Solving the normal equations in general terms,

$$x = 0\ 1193\ [al] + 0\ 0224\ [bl] + 0\ 0616\ [cl]$$
$$y = 0\ 0224\ [al] + 0\ 1049\ [bl] + 0\ 0384\ [cl]$$
$$z = 0\ 0616\ [al] + 0\ 0384\ [bl] + 0\ 1057\ [cl]$$

Hence

$$u_x = [aa] = 0\ 1193$$
$$u_y = [\beta\beta] = 0\ 1049$$
$$u_z = [\gamma\gamma] = 0\ 1057$$

as found in Art 95

98 In deducing the formulas for the precision of the adjusted values in a series of normal equations we have, for convenience in writing, taken the observation equations to be reduced to weight unity, and the normal equations, consequently, to be of the form

$$[aa]x + [ab]y + \qquad = [al]$$
$$[ab]x + [bb]y + \qquad = [bl] \qquad (1)$$

The formulas with the weight symbols introduced, corresponding to those found in the preceding articles, are easily derived from them by writing $a \sqrt{p}$, $b \sqrt{p}$, $l \sqrt{p}$, for a, b, l, and $\sigma \sqrt{u}$, $\beta \sqrt{u}$,   for σ, β,   respectively   (See Art 58)

Thus, for example, from the normal equations

$$[paa]x + [pab]y + \quad\quad = [pal\,|$$
$$[pab]x + [pbb]y + \quad\quad = |pbl\,| \quad\quad (2)$$

we should have

$$x = [\sigma l] = [u\sigma a][pal] + [u a\beta]|\,pbl\,| + $$
$$y = [\beta l] = [u\sigma\beta][pal] + [u\beta\beta]|\,pbl\,| + \quad\quad (3)$$

and by equating coefficients of l_1, l_2,   in the first expression,

$$[u\sigma a]a_1 + |u\sigma\beta|b_1 + \quad\quad = u_1\sigma_1$$
$$[u\sigma\beta]a_1 + [u\beta\beta]b_1 + \quad\quad = u_1\beta_1 \quad\quad (4)$$

99 To Find the m. s. e. μ of a Single Observation. —If the errors $\varDelta$ were known—that is, if the n observation equations were

$$a_1 x_0 + b_1 y_0 + \quad\quad - l_1 = \varDelta_1$$
$$a_2 x_0 + b_2 y_0 + \quad\quad - l_2 = \varDelta_2 \quad\quad (1)$$

where x_0, y_0,   are the true values of the unknowns —we should have at once

$$\mu^2 = \frac{[\varDelta\varDelta\,|}{n}$$

But we have only the residuals v with the observation equations

$$a_1 x + b_1 y + \quad\quad - l_1 = v_1$$
$$a_2 x + b_2 y + \quad\quad - l_2 = v_2 \quad\quad (2)$$

wheie x, y, aie the most probable values ol the un-
knowns We must, theielore, express $[\varDelta\varDelta]$ in terms of the
residuals v in ordei to find μ

Fiom the two sets of equations, by subtracting in pairs,

$$\varDelta_1 = v_1 + a_1(r_0 - x) + b_1(y_0 - y) + \ .$$
$$\varDelta_2 = v_2 + a_2(r_0 - x) + b_2(y_0 - y) + \qquad (3)$$

Now, taking the m s e μ_x, μ_y, to be the eiiois ol
x, y, , that is, to be equal to $x_0 - r$, $y_0 - y$, . , we
have fiom Eq 4, Art 93,

$$x_0 - x = \mu \sqrt{[\alpha\alpha]}, \quad y_0 - y = \mu \sqrt{[\beta\beta]}, \ .$$

and theiefore

$$\varDelta_1 = v_1 + \mu(a_1 \sqrt{[\alpha\alpha]} + b_1 \sqrt{[\beta\beta]} + \qquad)$$
$$\varDelta_2 = v_2 + \mu(a_2 \sqrt{[\alpha\alpha]} + b_2 \sqrt{[\beta\beta]} + \qquad)$$

Squaiing, adding, and attending to equations 10, Ait 93, we
have approximately, n_s being the number of unknowns,

$$[\varDelta\varDelta] = [vv] + n_s\mu^2 \qquad (4)$$

Putting $[\varDelta\varDelta] = n\mu^2$, theie iesults

$$\mu^2 = \frac{[vv]}{n - n_s} \qquad (5)$$

the expression iequiied

Reasoning as in Peters' formula, Ait 47, we easily de-
duce from (4)

$$\mu = 1\ 2533 \frac{[v]}{\sqrt{n(n - n_s)}} \qquad (6)$$

which is known as Luroth's formula (*Astion Nachi* , 1740)

When $n_s = 1$, equations 5 and 6 reduce to Bessel's and
Peteis' foimulas respectively (Aits 43, 47)

100 *Methods of Computing* $[vv]$ — (a) The oidinaiy method
is to substitute the values of the unknowns found fiom the
solution of the normal equations in the obseivation equa-

25

tions, and thence find v_1, v_2,   The sum of the squares of
these residuals will give $[vv]$

The residuals having to be found, for the purpose of
testing the quality of the work, this method of computing
$[vv]$ is on the whole as short as any

As checks on the values of $[vv]$ found in this way the
following are of value

(b) If we multiply each observation equation by its a,
and take the sum of the products, then, remembering that
$[av] = 0$, $[bv] = 0$,   , we find

$$[ca]   [al]$$

(c) If we multiply each of the observation equations by
its l and take the sum of the products,

$$[al]\,x\ +\ [bl]\,y\ +\ \qquad [ll]\ -\ [vl]$$
$$[vv]$$

(d) We have for two unknowns, x and y,

$[vv] = [(ax + by - l)^2]$

$$=\ [aa]x^2\ +\ 2[ab]xy\ +\ [bb]y^2\ -\ 2[al]x\ -\ 2[bl]y\ +\ [ll]$$

$$=[aa]\left(x\ +\ \frac{[ab]}{[aa]}y\ -\ \frac{[al]}{[aa]}\right)^2\ +\ \left([bb]\ -\ \frac{[ab]}{[aa]}[ab]\right)y^2$$
$$\left(2[bl]\ -\ \frac{[ab]}{[aa]}[al]\right)y\ +\ [ll]\ -\ \frac{[al]^2}{[aa]}$$

$$=[aa]\left(x\ +\ \frac{[ab]}{[aa]}y\ -\ \frac{[al]}{[aa]}\right)^2\ +\ [bb\,1]y^2\ -\ 2[bl\,1]y\ +\ [ll\,1]$$

$$=[aa]\left(x\ +\ \frac{[ab]}{[aa]}y\ -\ \frac{[al]}{[aa]}\right)\ +\ [bb\,1]\left(y\ -\ \frac{[bl\,1]}{[bb\,1]}\right)$$
$$+\ [ll\,1]\ -\ \frac{[bl\,1]}{[bb\,1]}$$

$$=[aa]\left(x\ +\ \frac{[ab]}{[aa]}y\ -\ \frac{[al]}{[aa]}\right)^2\ +\ [bb\,1]\left(y\ -\ \frac{[bl\,1]}{[bb\,1]}\right)^2\ +\ [ll.2]$$

and generally for m unknowns

$$[vv] = [aa] + [bv] + \ldots + [lz]$$

$$= [aa]\left(1 + \frac{[av]}{[aa]} + \ldots + \frac{[al]}{[aa]}\right)^2$$

$$+ [bb.1]\left(v + \frac{[bv.1]}{[bb.1]} + \ldots + \frac{[bl.1]}{[bb.1]}\right) + \ldots + [ll.m]$$

Now, from (6), Art. 76, the coefficients of $[aa], [bb.1], \ldots$ are each equal to zero. Hence

$$[vv] = [ll.m]$$

$$= [ll] - \frac{[al]^2}{[aa]} - \frac{[bl.1]^2}{[bb.1]} - \frac{[cl.2]^2}{[cc.2]} - \ldots$$

This expression was first given by Gauss (*De Elementis Ellipticis Pallades*, Art. 13). Its form suggests that if we add an extra column to the normal equations, as shown in the following scheme, we shall find vv at the same time as the last unknown. This is analytically very elegant, and, as the check (see Art. 3) can be carried with this column through the solution of the normal equations, it may be used for finding $[vv]$ if one is computing alone. Only one extra term $[ll]$ has to be computed while forming the normal equations.

The scheme is as follows.

Ex 1 To find the m s e of the adjusted values of the unknowns found in Ex 4, Art° 83

The first step is to find $[pvv]$ This we shall do in the four ways indicated

(a)

v	p	pvv
$- 0\ 05$	5	$0\ 01$
$- 0\ 36$	7	$0\ 91$
$+ 0\ 68$	4	$1\ 85$
$- 0\ 03$	7	$0\ 01$
$- 0\ 62$	4	$1\ 54$
		$4\ 32 = [pvv]$

(b)

$$p_1 l_1 v_1 = 0$$
$$p_2 l_2 v_2 = 0$$
$$p_3 l_3 v_3 = 0$$
$$p_4 l_4 v_4 = 7 \times 0\ 76 \times - 0\ 03 = - 0\ 16$$
$$p_5 l_5 v_5 = 4 \times 1\ 66 \times - 0\ 62 = - 4\ 12$$
$$- 4\ 28 = + [pvl] = - [pvv]$$

(c)

$$[pal]x = - \ 5\ 32 \times - 0\ 05 = 0\ 27$$
$$[pbl]y = - \ 6\ 64 \times - 0\ 36 = 2\ 39$$
$$[pcl]z = + 11\ 96 \times + 0\ 68 = 8\ 13$$
$$10\ 79$$
$$p_1 l_1 l_1 = 0$$
$$p_2 l_2 l_2 = 0$$
$$p_3 l_3 l_3 = 0$$
$$p_4 l_4 l_4 = 7 \times (0\ 76)^2 \qquad = \ 4\ 04$$
$$p_5 l_5 l_5 = 4 \times (1\ 66)^2 \qquad = 11\ 02$$
$$15\ 06$$
$$4\ 27 \ = [pvv]$$

(d) We find $[pll] = 15\ 06$

The solution of the normal equations, with the extra column for $[hl]$ added would be according to the foregoing scheme.

Ex 3 Show that

$$[vv] = [\Delta\Delta] - \frac{[a\Delta]^2}{[aa]} - \left[\frac{b\Delta\ 1}{bb\ 1}\right]^0 - \quad \cdot$$

[Form the normal equations from equations 3, Art 99, and

$$[aa](x_0 - x) + [ab](y_0 - y) + \qquad = [a\Delta]$$
$$[ab](x_0 - x) + [bb](y_0 - 1) + \qquad = [b\Delta]$$
$$\cdot \qquad\qquad \cdot$$

since $[av] = [bv] = \qquad = 0$

Hence from Art 100

$$[\Delta\Delta] = [vv] + \frac{[a\Delta]^2}{[aa]} + \frac{[b\Delta\ 1]^2}{[bb\ 1]} + \qquad\qquad |$$

Ex 4 From the equation

$$[al]x + [bl]y + \qquad - [ll] = - [vv]$$

and

$$x = \frac{[al]}{[aa]} + \frac{[bl\ 1]}{[bb\ 1]}R_1 + \frac{[cl\ 2]}{[cc\ 2]}R_2 +$$

$$y = \qquad \frac{[bl\ 1]}{[bb\ 1]} \qquad + \frac{[cl\ 2]}{[cc\ 2]}S_2 +$$

$$\cdot \qquad \cdot \qquad\qquad \cdot$$

deduce

$$[vv] = [ll] - \frac{[al]^2}{[aa]} - \left[\frac{bl\ 1}{bb\ 1}\right]^0 -$$

Ex 5   Prove that $[av] = [\beta v] = \qquad = 0$

Ex 6   From

$$[vv] = [\Delta\Delta] - \frac{[a\Delta]^2}{[aa]} - \frac{[b\Delta\ 1]^2}{[bb\ 1]} -$$

deduce the formula

$$\mu^2 = \frac{[vv]}{n - n_x}$$

101. To Find the Precision of any Function of the Adjusted Values X, Y, . This is the more general case of the problem just discussed. The method of solution is

First, to find μ, the m. s. e. of an observation of weight unity, and next p, the weight of the function, whence the m. s. e. of the function is given by

$$\mu \sqrt{u_f}$$

where u_f is the reciprocal of p.

The value of μ is computed from (5) or (6), Art. 99.

Next, to find u_f. Let the function be

$$f = f(X, Y, \ldots) \qquad (1)$$

in which X, Y, . . . are functions of the independently observed quantities H, H, . . . H_n.

By differentiation

$$df = \frac{\partial f}{\partial H} dH + \frac{\partial f}{\partial H} dH + \cdots + \frac{\partial f}{\partial H_n} dH_n$$

and therefore, since H, H, . . . are independent,

$$u_f = \left(\frac{\partial f}{\partial H}\right) u + \left(\frac{\partial f}{\partial H}\right) u + \cdots + \left(\frac{\partial f}{\partial H_n}\right) u_n \qquad (2)$$

where u, u, . . . u_n are the reciprocals of the weights of the observed quantities.

(a) Instead, however, of using the general formula directly, it is, in general, a great saving of labor to compute from a modified form as follows, by means of which much of the work already done in solving the normal equations may be utilized.

Reducing the function to the linear form, we have, adopting the notation of Art. 81,

$$F = f(X' + x, \ Y' + y, \quad . \quad)$$
$$= f(X', \ Y', \qquad) + \frac{\delta F}{\delta X'} x + \frac{\delta F}{\delta Y'} y +$$

or, as it may be written,

$$dF = G_1 x + G_2 y + \quad . \tag{3}$$

Now, since x, y, are not independent, but are connected by the equations

$$[aa]x + [ab]y + \quad = [al]$$
$$[ab]x + [bb]y + \quad = [bl]$$

we must get rid of this entanglement by expressing these quantities x, y, in terms of l_1, l_2, , which are independent of each other   From Art. 93 we may write

$$x = [al], \ y = [\beta l],$$

where $\sigma_1, \sigma_2,$, $\beta_1, \beta_2,$,   are functions of a_1, b_1,   , a_2, b_2,   ,   Hence, substituting in (3)

$$dF = (G_1\sigma_1 + G_2\beta_1 + \quad)l_1 + (G_1\sigma_2 + G_2\beta_2 + \quad . \quad)l_2 + \quad .$$

and, therefore (Arts 19, 56), since the observation equations have been reduced to weight unity,

$$u_F = (G_1\sigma_1 + G_2\beta_1 + \quad)^2 + (G_1\sigma_2 + G_2\beta_2 + \quad)^2 + \quad .$$
$$= G_1^2 [\sigma\sigma] + 2G_1G_2 [\sigma\beta] + \quad .$$
$$+ \ G_2^2 \ [\beta\beta] + \tag{4}$$
$$+ \quad .$$

where $[\sigma\sigma]$, $[\sigma\beta]$ may be found in the manner indicated in Arts 94, 96, or 97   Hence U is known
   (b) Eq 4 may be written

$$u_F = G_1 Q_1 + G_2 Q_2 + \quad + G_n Q_n \tag{5}$$

where

$$Q_1 = [aa]G_1 + \lceil a_1\beta\rceil G_1 +$$
$$Q_2 = \lceil a\beta\rceil G_1 + \lceil \beta_1\beta\rceil G_2 + \qquad (6)$$
·

that is, where (see Eq 1, Art 97)

$$G_1 = |aa|Q_1 + \lceil ab\rceil Q_1 +$$
$$G_2 = [ab|Q_2 + [bb|Q_2 + \qquad (7)$$

Hence the weight of a function

$$G_1 x + G_2 y +$$

of several independent unknowns x, y,   is found from

$$u_F = |GQ|$$

where Q_1, Q_2,   satisfy the equations

$$|aa|Q_1 + |ab|Q_2 + \qquad = G_1$$
$$|ab|Q_1 + |bb|Q_2 + \qquad = G_2$$
·   ·

Therefore we conclude that, *if in a series of observation equations the values of the unknowns x, y,   are required, as well as their weights or the weight of any function of them, these results can be found at one time by making a solution of the normal equations for finding x, y,   in general terms, and then substituting for $|al|$, $|bl|$,   their numerical values on the one hand and the values of G_1, G_2,   on the other*

(c) This result may be stated in other forms   Thus from Eq 4, by substituting for $|aa|$, $|a\beta|$,   their values from Art 96, or by substituting for Q_1, Q_2,   their values in (6) as expressed in Art 96, we have, after a simple reduction,

$$u_I = \frac{G_1^2}{[aa]} + \frac{(G_1 R_1 + G_2)^2}{|bb\,1|} + \frac{(G_1 R_1 + G_2 S + G_3)^2}{|cc\,2|} + \qquad (8)$$

Comparing this expression with (d), Art 100, it is evident that the several terms are such as would result from the

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following elimination (Ex three unknowns) by finding the products of the quantities bracketed

x	y	z	
$\begin{bmatrix}aa\\ab\\ac\end{bmatrix}$	$\begin{bmatrix}ab\\bb\\bc\end{bmatrix}$	$\begin{bmatrix}ac\\bc\\cc\end{bmatrix}$	$\begin{cases}G_1\\G_2\\G_3\end{cases}$
1	$\begin{array}{l}[ab]\\ \overline{[aa]}\\ [bb\ 1]\\ [bc\ 1]\end{array}$	$\begin{array}{l}[ac]\\ [aa]\\ [bc\ 1]\\ [cc\ 1]\end{array}$	$\begin{array}{l}G_1\\ [aa]\\ G_1R_1 + G_2\end{array}$
	1	$\begin{array}{l}[bc\ 1]\\ [bb\ 1]\\ [cc\ 2]\end{array}$	$\begin{array}{l}G_1R_1 + G_2\\ [bb\ 1]\\ G_1R_2 + G_2S_2 + G_3\end{array}$
		1	$\begin{array}{c}G_1R_3 + G_2S_3 + G_3\\ [cc\ 2]\end{array}$

(d) The expression (8) for u_F may be easily transformed into

$$k_0{}^2[aa] + k_1{}^2[bb\ 1] + k_2{}^2[cc\ 2] +$$

where

$$[aa]k_0 \qquad\qquad = -G_1$$
$$[ab]k_0 + [bb\ 1]k_1 \qquad = -G_2$$
$$[ac]k_0 + [bc\ 1]k_1 + [cc\ 2]k_2 = -G_3$$

Circumstances must decide which of the four forms given should be chosen in any special case. A machine can be used to the best advantage with the second and third forms. The third form is also convenient when the weights alone are required, without the values of the unknowns, and the second when the values of the unknowns can be found by an easier method of solution than the Gaussian method of substitution

Ex 1 In Ex 1, Art 83, it is required to find the m s e of the angle *PSB*

The function is

$$dF = -x + z$$

First Solution From Eq 4,

$$u_F = [a\alpha] - 2[a\gamma] + [\gamma\gamma]$$
$$= 0\ 1193 - 2 \times 0\ 0616 + 0\ 1057\ \text{(from Ex , Art 97)}$$
$$= 0\ 1018$$

Second Solution From equations 7,

$$+ 12Q_1 \qquad\qquad - 7Q_1 = -1$$
$$+ 11Q_2 - 4Q_1 = \quad 0$$
$$- 7Q_1 - 4Q_1 + 15Q_1 = +1$$

Hence

$$Q_1 = -0\ 0577,\ Q_1 = +0\ 0440$$

and

$$u_F = -1 \times -0\ 0577 + 1 \times 0\ 0440$$
$$= 0\ 1017$$

Third Solution Add the extra column *G* to the solution of the normal equations, which would give the scheme

x	y	z	G
+ 12			
	+ 11	− 7	− 1
		− 4	0
		+ 15	+ 1
+ 1			
	+ 11	− 0 5833	− 0 0833
		−4	0
		+ 10 9169	+ 0 4169
	+ 1	− 0 3636	0
		+ 9 1625	+ 0 4169
	•	+ 1	+ 0 0441

Hence

$$u_F = -1 \times -0\ 0833 + 0\ 1169 \times 0\ 0441$$
$$= 0\ 1017$$

Fourth Solution

$$12k_0 \qquad\qquad\qquad 1\ 1$$
$$11k_1 \qquad\qquad\qquad 0$$
$$7k_0 \quad 4k_1 + 0\ 10^{...}, \qquad 1$$
$$k_0 \quad 0\ 0543,\ k_1 \quad 0, 7 \qquad 00440$$
$$u_k \quad (0\ 0533)'\times 12 + (0\ 0440) \quad 0\ 46^{...}$$
$$0\ 1016$$

Also

$$\mu_k' \quad 1 \quad 1/4\ 0\ 10^{..}$$
$$\sigma''\ 17$$

Ex 2 Given the observation equations

$$a_1 x + b_1 v \quad l_1$$
$$a_2 x + b_2 v \quad l_2$$

$$a_n x + b_n v \quad l_n$$

to find the weight of $x + yv$

[The normal equations are

$$[aa]x + [ab]v \quad [al]$$
$$[ab]x + [bb]v \quad [bl]$$

$$u_k \quad \frac{1}{[aa][bb\ 1]}\{[bb]/^{..} \quad [ab][l] + [aa]; \{ \quad$$

102 *To find the average value of the ratio of the weight of the observed value of a quantity to that of its adjusted value, in a system of independently observed quantities.*

The adjusted value of the first observed quantity M_1 is $M_1 + v_1$. From Art 81 it follows that the weight of $M_1 + v_1$ is the same as the weight of $l_1 + v_1$. Now,

$$l_1 + v_1 \quad a_1 x + b_1 v + \ldots \tag{1}$$

Hence if P_1 is the weight of the adjusted value $M_1 + v_1$, that is, is the weight of the function $a_1 x + b_1 v + \ldots$, and $p_1, p_2, \ldots$ are the weights of $l_1, l_2, \ldots$, we have

$$\frac{1}{P_1} \quad a_1 Q_1 + b_1 Q_2 + \ldots \tag{2}$$

where (see Eq 4, Art. 98, (b) Art. 101)

$$Q_1 = [uua|a_1 + [ua\beta|b_1 + . \qquad u_1 u_1$$
$$Q_2 = [ua\beta|a_1 + [u\beta\beta|b_1 + \ldots \qquad u_1 \beta_1 \tag{3}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Therefore by substitution of $Q_1, Q_2, \quad$ in (2),

$$\frac{p_1}{p_1} \quad a_1 a_1 \mid b_1 \beta_1 + \mid$$

Similarly

$$\frac{p}{p} \quad a\, a_2 \mid b_2 \beta \mid$$

. .

Hence by addition

$$\left[\frac{p}{p}\right] \quad \mid aa \mid \mid \mid b_1 \beta \mid \mid \qquad \text{to } n_u \text{ terms}$$

n_u, the number of independent unknowns.

This result may be very readily derived directly as follows In (1) put $\mid a \mid$ for x, $\mid \beta \mid$ for y, . , and we have

$$l_1 \mid c_1 \quad (a_1 a_2 \mid b_1 \beta_1 \mid \qquad)l_1 \mid (a_1 a_2 \mid b_1 \beta_1 \mid . \qquad)l_2 \mid$$

Hence, since $l_1, l_2, \quad$. are independent,

$$\frac{1}{p_1} \quad (a_1 a_1 \mid b_1 \beta_1 \mid \qquad . \,)^2 \frac{1}{p_1} \mid (a_1 a_1 \mid b_1 \beta_1 \mid \qquad)^2 \frac{1}{p_2} \mid \quad .$$

$$a_1 \} \mid aaa \mid a_1 \mid \mid \mid aa_1 \beta \mid b_1 \mid \qquad \}$$

$$\mid b_1 \} \mid aa_1 \beta \mid a_1 \mid \mid \mid a_1 \beta \beta \mid b_1 \mid \qquad \}$$

.

$$n_1 (a_1 a_1 \mid b_1 \beta_1 \mid \qquad) \qquad \qquad (4)$$

as before.

103 From the principle just proved we may derive a proof of the formula found in Art 99 for the m s e of an observation of weight unity in a series of n observations involving n_i independent unknowns It is analogous to the proof given for the m s e of a single observation in a series of n observations of one unknown

The errors of the observed values $H_1, H_2, \ldots H_n$ are $J_1, J_2, \ldots J_n$, and the errors of the most probable values $V_1, V_2, \ldots V_n$ may be assumed to be $\mu_1, \mu_2, \ldots \mu_n$ respectively Hence since

$$ \Delta I = V - v $$

we have

$$ J_1 = \mu_1 \quad v_1 $$
$$ J = \mu_1 \quad v $$

$$ J_n = \mu_n - v_n $$

and, therefore,

$$ |pJJ| \quad |p\mu_1\mu_1| \quad |pvv| \tag{1} $$

Again, since P_1 is the weight of I_1,

$$ \mu_{v_1} = \frac{\mu}{P_1} $$

where μ is the m s e of an observation of weight unity Hence

$$ p_1\mu_{11} \quad \frac{p_1}{P_1}\mu $$

$$ p_1\mu_1 \quad \frac{p}{P}\mu $$

$$ p_n\mu_{v_n} \quad \frac{p_n}{P_n}\mu $$

By addition,

$$ |p\mu_1\mu_1| - \left[\frac{p}{P}\right]\mu^2 $$
$$ - n_i\mu^2 \tag{2} $$

But by definition

$$\mu = \frac{|\,p\underline{\underline{\underline{l}}}\,|}{n}$$

Substituting in (1), we have finally

$$\mu = \frac{|\,p\upsilon\upsilon\,|}{n - n_{l}}$$

the result required

Miscellaneous Examples and Artifices of Elimination

In this section will be discussed several problems of importance, and also certain artifices of elimination which may often be employed with advantage in the solution of observation equations

104 The labor of solving and finding the values of the unknowns may often be shortened by taking advantage of some principle inherent in the form of the observation equations themselves For example, if we have a series of observation equations containing two unknowns, and of which the coefficient of the first unknown is unity, instead of solving in the usual way we may reduce the observation equations to equations containing the second unknown only, and thus solve more readily

Given

$$x + b_{1}y = l_{1} \quad \text{weight } p_{1}$$
$$x + b\,y = l \qquad " \qquad p$$

Forming the normal equations in the usual way, we have

$$|\,p\,|x + |\,pb\,|\,y = |\,pl\,|$$
$$|\,pb\,|x + |\,pb\,|\,y = |\,pbl\,|$$

whence eliminating x,

$$y\{|\,pb\,| - |\,p\,||\,pb\,|\} = |\,pl\,||\,pb\,| - |\,p\,||\,pbl\,|$$

Now, if the first normal equation is divided by $|\,p\,|$, so that

$$x + \frac{|\,pb\,|}{|\,p\,|}\,y = \frac{|\,pl\,|}{|\,p\,|}$$

and from this equation each of the observation equations in succession is subtracted, there result the equations

$$\left(\frac{|pb|}{|p|} - b_1\right) v = \frac{|pl|}{|p|} - l_1 \quad \text{weight } p$$

$$\left(\frac{|pb|}{|p|} - b\right) v - \frac{|pl|}{|p|} - l \qquad \qquad p$$

The normal equation for finding v from these equations is

$$\{|p||pb| - |pb|\}v - |p||pbl|, \qquad p$$

the same as results from the elimination of in the normal equations

This process is specially convenient if the original observation equations are numerous and the coefficients b, and the terms $l_1, l,$ large and not widely different

Ex To solve the equations in Ex , Art
The mean of the equations is

$$v - 0.9, l - l - a,$$

Subtract each equation from this mean equation and

$$\begin{aligned} 1 - 0.6 v & - 1 - a_1 \\ 1 - 0.1 v & - a_2 \\ 1 - 0.20 v & - 1 - a_3 \\ 0.01 v & - a \\ 0.25 v & - 1 - a_4 \\ 1.08 v & - a \end{aligned}$$

The normal equation formed from these equations is

$$1 - 1.91 v - 0.7$$

and $1 - 0.01$

Substitute for v its value in the mean equation, and

$$v - l - a$$

105   Again, we may take advantage of the presence in the problem of some arbitrary quantity to which a convenient value may be assigned   Thus, to find the difference of the coefficients of expansion of two standards l and l'

from observed differences of length at certain fixed temperatures

Let ι = the excess in length of A over B at an arbitrary temperature t_0

$\quad y$ = the excess of the coefficient of expansion of A over that of B,

l_1, l_2,   = the observed differences in length at temperatures t_1, t_2,   and whose weights are p_1, p ,

We have then the observation equations

$$\iota + (t_1 - t_0)y - l_1 = r_1 \quad \text{weight } p_1$$
$$\iota + (t - t_0)y - l_2 = r' \quad \text{weight } p \tag{1}$$

and the normal equations

$$|p|\iota + \{|pt| - |p|t_0\}y = |pl|$$
$$\{|pt| - |p|t_0\}\iota + |p(t - t_0)^2|y = |(t - t_0)pl| \tag{2}$$

As the value of t_0 is arbitrary, the normal equations will be simplified by taking it equal to the weighted mean of the temperatures, that is,

$$t_0 = \frac{|pt|}{|p|} \tag{3}$$

and they will then become

$$|p|\iota \qquad\quad = |pl|$$
$$|p(t - t_0)|y = |(t - t_0)pl| \tag{4}$$

from which ι and y are found at once, with their weights at the mean temperature t_0

If the values of l are numerically large it will lessen the labor of finding the value of y if the mean value of ι found from

$$|p|\iota = |pl|$$

is substituted in the observation equations before the normal equation in y is formed   We should then have

$$|p(t - t_0)|y = |p(t - t_0)(l - \iota)|$$

from which to find y

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It is evident that the value of r found in this way is the same as before   For

$$[p(t - t_0)(l - 1)] - [p(t - t_0)l] + 1[p l] - [p l l]$$
$$= -[p(t - t_0)l]$$

since the coefficient of 1 is equal to zero from Eq   3

The quantity $l - 1$ comes in very conveniently in computing the residuals v in finding the precision

The precision.—If n is the number of observations, the number of unknowns being 3, we have for the m s e μ of a single observation

$$\mu = \sqrt{\frac{[p v v]}{n}},$$

and

$$\mu_x = \frac{\mu}{\sqrt{[p]}}$$

$$\mu_y = \frac{\mu}{\sqrt{[p(t - t_0)^2]}}$$

The length at any temperature t' is

$$x + (t' - t_0) y$$

and its m s e $\mu_{x'}$ is found from

$$\mu_{x'} = \mu_x^2 + (t' - t_0) \mu_y$$

$$= \frac{\mu}{\sqrt{[p]}} + \frac{(t' - t_0)}{\sqrt{[p(t - t_0)]}} \mu$$

The weight is greatest when $\mu_{x'}$ is least, that is, when

$$t' - t_0 = \frac{[p t]}{[p]}$$

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153 The following were among the observations made for the determination of the difference of length between the Lake Survey Standard Bar and Yard, and also for the difference between their coefficients of expansion   The unit is $\frac{1}{100000}$ inch

Required the difference of length at 62° Fahr and at any other temperature t

Date	Observed temp (t)	Bar − Yard (l)	Weight (p)
1872, March 5	24·7	791	5
" 14	37·1	811	1
" 26	61°·7	833	6
April 4	49·3	820	6
" 12	66·5	847	8
" 20	71°·5	849	8

Let x = the most probable difference between Bar and Yard at 62° Fahr,
y = the most probable difference between coefficients of expansion of Bar and Yard

The observation equations will be of the form

$$x + (t - 62)y - l = v$$

The computation is arranged in tabular form as follows

p	pt	pl	$t - t_0$	$l - x$
5	123·5	3955	− 32·2	− 40
1	37·1	811	− 19·8	− 20
6	370·2	4998	+ 4·8	+ 2
6	295·8	4920	− 7·6	− 11
8	531·4	6776	+ 9·9	+ 16
8	572·0	6792	+ 14·6	+ 18
34	1933·0	26252		
	$t_0 = 56·9$	$x = 831$		

$p(t - t_0)^2$	$p(t - t_0)(l - x)$	$y(t - t_0)$	v	pvv
518·20	644·0	− 40·6	− 0·6	1·8
392·04	396·0	− 24·9	− 4·9	24·0
138·24	57·6	+ 6·0	+ 4·0	96·0
346·56	501·6	− 9·6	+ 1·4	11·8
784·08	1267·2	+ 12·5	− 3·5	98·0
1705·28	2102·4	+ 18·4	+ 0·4	1·3
8550·40	10764·5			232·9
	$y = 1·26$			

$$\mu = \sqrt{\frac{232\ 9}{6-2}} = 7\ 6$$

$$\mu_x = \frac{7\ 6}{\sqrt{34}} = 1\ 3$$

$$\mu_y = \frac{7\ 6}{\sqrt{8550\ 40}} = 0\ 1$$

Value of $x = 831$ at $56\ 9$
$$\frac{6\ 4}{= 837\ 4}\qquad \frac{5\ '\ 1}{62°\ 0}$$

$$\mu_{61}{}^2 = (1\ 3)^2 + (5\ 1)^2 \times (0\ 1)° = 1\ 9$$

These values may be checked by computing by the ordinary process

The application of the preceding method to the solution of this problem is due to Mr E S Wheeler

106   It often happens that of two series of observed quantities one is a function of the other, and that, therefore, the observed quantities may be regarded as co-ordinates of points in the curve which represents the relation connecting them   From the plot of the points the general form of the curve can be judged, and then the special form which satisfies the observations may be found as follows

Let us investigate the important practical case where the co-ordinates have been equally well measured and the curve is a straight line, both co-ordinates being regarded fallible

Let x_1, y_1 x_2, y_2 x_n, y_n be the observed values of the co-ordinates, and X_1, Y_1 X_2, Y_2 X_n, Y_n their most probable values

The conditions to be satisfied are

$$Y_1 = aX_1 + b$$
$$Y_2 = aX_2 + b \qquad\qquad (1)$$
$$\cdot\ \cdot\ \cdot\ \cdot$$
$$Y_n = aX_n + b$$

where a and b are constants, to be found from the observations.   Also

$$(X_1 - x_1)^2 + (Y_1 - y_1)^2 + \qquad = a\ \text{min} \qquad (2)$$

Eliminate the Y's by substituting from equations 1 in (2), and then

$$(X_1 - x_1)^2 + (aX_1 + b - y_1)^2 + \qquad = a\ \text{minimum}$$

Differentiate this equation with respect to the independent variables X_1, X_2, X_n, and

$$X_1 - \lambda_1 + a(aX_1 + b - y_1) = 0$$
$$X_2 - \lambda_2 + a(aX_2 + b - y_2) = 0$$

$$X_n - \lambda_n + a(aX_n + b - y_n) = 0$$

from which equations X_1, X_2, X_n may be expressed in terms of a, b and the known quantities λ_1, y_1, , x_n, y_n

Again, differentiate with respect to a, b, and we find

$$a[X^2] + b[X] = [Xy]$$
$$a[X] + bn = [y]$$

Hence a and b are known, and therefore the equation of the straight line is known

The same mode of reasoning may be applied to the passing of a curve of a specified form through n points, whether their co-ordinates have been equally well measured or not

The problem of passing a line which should deviate as little as possible from the positions of several given points was discussed by Lambert [x] as long ago as 1765, thirty years before Gauss first made use of the method of least squares Lambert reasoned that if the line is supposed drawn the points should deviate as much on the one side as on the other, and hence that the line would pass through the centre of gravity of the points

107. **Pothenot's Problem.**—In topographical work the three-point problem, usually called Pothenot's problem, is of great importance It may be stated as follows

Let P_1, P_2, P_3 be three points whose positions are known — as, for example, by their co-ordinates with reference to known axes—and P a point at which the angles $P_1PP_2 = \varphi_1$, $P_2PP_3 = \varphi_2$, are

Fig 10

[x] *Beyträge zum Gebrauche der Mathematik* Berlin, 1765
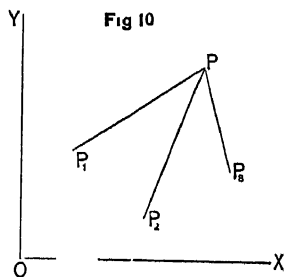
observed   It is required from these data to determine the position of P

The geometrical solution is simple   Plot the points P_1, P_2, P_3 Describe on the line $P_1'P_2'$ a circle containing an angle equal to φ_1, and on $P_2'P_3$ a circle containing an angle equal to φ_2 The intersection of these circles will be the point required

A solution much used in practice—as, for example, in plotting soundings—is to lay off on a piece of tracing-linen two adjacent angles equal to φ_1, φ_2, and then move this figure over the map on which the points P_1, P_2, P_3 are plotted until the directions PP_1, PP_2, PP_3 lie over the points P_1, P_2, P_3 respectively   The position of P is then pricked through

So far as the plotting of soundings is concerned, the above is in general sufficient, but in the location of important secondary points with reference to known primary points greater precision is necessary

The position of P may be computed trigonometrically as follows   Since the positions of the points P_1, P_2, P_3 are known, the lengths of the lines P_1P_2, P_2P_3, P_1P_3 are known, and, therefore, the angle $P_1P_2P_3$ can be found   Call its value β

Denote the angles PP_1P_2, PP_3P_2 by a_1, a_2 respectively.

The sum of the angles of each of the triangles PP_1P_2, PP_1P_3 being 180°, we have

$$\beta + a_1 + \varphi_1 + a_2 + \varphi_2 = 360°$$

and, therefore,

$$a_1 + a_2 = 360° - \varphi_1 - \varphi_2 - \beta$$

a known quantity

Again,

$$\frac{PP_2}{\sin a_1} = \frac{P_1P_2}{\sin \varphi_1} \qquad \frac{PP_2}{\sin a_2} = \frac{P_2P_3}{\sin \varphi_2}$$

$$\frac{\sin a_2}{\sin a_1} = \frac{P_1P_2 \sin \varphi_2}{P_2P_3 \sin \varphi_1} = \tan \gamma \text{ suppose,}$$

and

$$\frac{\sin a_1 + \sin a_2}{\sin a_1 - \sin a_2} = \frac{1 + \tan \gamma}{1 - \tan \gamma}$$

or

$$\cot \tfrac{1}{2} (a_1 - a_2) = \tan (\gamma + 45) \cot \tfrac{1}{2} (a_1 + a_2)$$

and, therefore, $a_1 - a_2$ is known   Combining with the value of $a_1 + a_2$, we find a_1 and a_2

Hence the position of P is completely determined

It often happens, as in the determination of the position of a light-house, for example, that more than three points are sighted at from the point occupied, and that there are more observations than are necessary to locate the point Its most probable position can consequently be found   We proceed to show the method of finding it

Let P_1, P_2,   be the known points in order of azimuth, as seen from the unknown point P X_1, Y_1 X_2, Y_2, the co-ordinates of P_1, P_2,   referred to some known system of rectangular axes, preferably the parallel and meridian at the point chosen as origin of co-ordinates θ_1, θ_2, the angles which the most probable positions of PP_1, PP_2,   . make with the axis of x θ_1', θ_2',   approximate values of these angles as computed from the co-ordinates, and ϑ_1, ϑ_2,   corrections to these approximate values so that

$$\theta_1 = \theta_1' + \vartheta_1, \ \theta_2 = \theta_2' + \vartheta_2,$$

φ_1, φ_2,   the angles observed at P between the directions PP_2, PP_3,   and the initial direction PP_1

X, Y,   the co-ordinates of P referred to the same axes as X_1, Y_1 X_2, Y_2,   Suppose X', Y' to be close approximations to the values of X, Y, found graphically, and x, y corrections to these values, so that

$$X = X' + x, \ Y = Y' + y,$$

Now,

$$\tan \theta_1 = \frac{Y - Y_1}{X - X_1}$$

or

$$\tan (\theta_1' + \theta_1) \qquad \} \qquad \} \, , \, | \, \, $$

Take logs of both members and expand, then

$$\log \tan \theta_1' + \delta_1 \theta_1 \quad \log (\} \, ' \quad \}_1) + \delta_1 + \log \langle \Lambda \quad \Lambda + \, \, , \, \, $$

where $\delta_1, \delta_2, \delta_3$ are tabular differences, as explained in Art

But

$$\tan \theta_1' \qquad \frac{\}''}{\backslash \, .} \quad \frac{\}_1'}{\backslash_1}, \text{ and is, therefore known}$$

Hence

$$\theta_1 \quad \frac{\delta}{\delta_1} \, r \quad \frac{\delta_1}{\delta_1}$$

$$a_1 + b_1 r \text{ suppose,}$$

or $\theta_1 \quad \theta_1' + a_1 + b_1$

Similarly

$$\theta_2 \quad \theta' + a_{11} + b \, r$$

Now, comparing computed and observed values of the angle $P_1 P P_u$,

$$\varphi_1 \quad \theta_1 \quad \theta$$

$$\theta_1' \quad \theta_1' + (a_1 \quad a_{11} + (b \quad b_{11}$$

Calling the quantity

$$\varphi_1 \quad (\theta_1' \quad \theta') \quad l_1$$

the observation equation may be written

$$(a_1 \quad a_{11}) i + (b_1 \quad b_{11}) r \quad l_1$$

An observation equation for each of the other angle $\varphi_{11}, \varphi_{111}$, observed at P may be formed in the same way

In the expansions above we have used log differences as being most convenient and as leading to results close enough in problems of this kind Five place tables are sufficient, indeed, in most cases four place are ample

On this problem consult Bessel, Zach's *Monatliche Correspondenz*, vol xxvii p 222, Gerling, *Pothenotsche Aufgabe*, Marburg, 1840, Gauss, *Astron Nachr*, vol 1 No 6, Schott, *C S Report*, 1864, App 13, Petzold, *Zeitschr fur Vermess*, vol xii p 227

.

Ex Given the rectangular co-ordinates of six known points, and the angles observed at the point P whose position is to be determined, as follows (*C S Report*, 1864, App 13)

$$X_1 = 1845\ 0 \qquad Y_1 = 5534\ 0 \qquad \varphi_1 = \ \ 61°\ 12'\ 10''$$
$$X_2 = 1485\ 0 \qquad Y_2 = 2486\ 7 \qquad \varphi_2 = \ \ 97°\ 48'\ 27''$$
$$X_3 = \ \ \ 0\ 0 \qquad Y_3 = \ \ \ 0\ 0 \qquad \varphi_3 = 195°\ 55'\ 56''$$
$$X_4 = 4418\ 2 \qquad Y_4 = 1416\ 7 \qquad \varphi_4 = 205°\ 35'\ 04''$$
$$X_5 = 6163\ 8 \qquad Y_5 = \ \ 398\ 1 \qquad \varphi_5 = 215°\ 53'\ 44''$$
$$X_6 = 5810\ 6 \qquad Y_6 = 1255\ 7$$

to find the co ordinates of P

By plotting the points we may take off the approximate values

$$X' = 3448 \qquad Y' = 2440$$

Following the form laid down in the preceding Art, and using five-place log tables (Art 7),

$$\log (3004 - v) \ \ = 3\ 49052\ \ - 14\ 0y$$
$$\log (- 1603 - x) = 3\ 20493n + 27\ 1x$$

$$\log \tan \Theta_1' + \delta_1 \vartheta_1 = 0\ 28559n - 27\ 1x - 14\ 0y$$

Now, $\Theta_1' = 117°\ 23'\ 19''$ and $\delta_1 = - 31$ for $1'$

Hence

$$\Theta_1 = 117°\ 23'\ 19'' + \ \ 52\ 5x + \ \ 27\ 1y$$

Similarly

$$\Theta_2 = 178°\ 38'\ 14'' + \ \ \ 2\ 5x + 103\ 6y$$
$$\Theta_4 = 215°\ 17'\ 07'' - \ \ 28\ 0x + \ \ 39\ 6y$$
$$\Theta_5 = 313°\ 28'\ 28'' - 103\ 2x - \ \ 97\ 8y$$
$$\Theta_6 = 323°\ 03'\ 45'' - \ \ 35\ 6x - \ \ 47\ 3y$$
$$\Theta_6 = 333°\ 22'\ 37'' - \ \ 35\ 6x - \ \ 71\ 0y$$

Subtracting Θ_1 from each of the values Θ_2, Θ_3, in succession, and comparing with the measured values φ_1, φ_2, , we have the observation equations

$$50\ 0x - \ \ 76\ 5y = 165$$
$$80\ 5x - \ \ 12\ 5y = 321$$
$$155\ 7x + 124\ 9y = 553$$
$$88\ 1x + \ \ 74\ 4y = 322$$
$$88\ 1x + \ \ 98\ 1v = 334$$

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Hence the normal equations

$$48746x + 29813\,y = 177960$$
$$29813x + 36767y = 100157$$

The solution gives

$$x = +3\,66 \quad \text{weight } 24572$$
$$y = 0\,02 \quad \text{weight } 18533$$

which added to the approximate values above give the final values

$$X = 3451\,66 \qquad Y = 2430\,98$$

Substitute for x, y in the observation equations, and we have the residuals

$$v_1 = +19\,5,\ v_2 = -27\,8,\ v_3 = +8\,1,\ v_4 = +10,\ v_5 = 11\,5$$

Hence, since $[vv] = 1402$,

$$\mu = \sqrt{\frac{1402}{5-2}} = 22$$

and

$$\mu_X = \frac{22}{\sqrt{24572}} = 0\,14$$

$$\mu_Y = \frac{22}{\sqrt{18533}} = 0\,16$$

CHAPTER V

108 We now take up the third division of the subject as
laid down in Art 39 So far the quantities we have dealt
with, whether directly observed or functions of the quan-
tities observed, have been independent of one another, but
if they are not independent of one another—that is, if they
must satisfy exactly certain relations that exist *à priori* and
are entirely separate from any relations demanded by ob-
servation—they are said to be *conditioned* by these relations

All problems relating to condition observations may be
solved by the rules laid down in the preceding chapters.

Let, with the usual notation, V_1, V_2, . V_n denote the
most probable values of n directly observed quantities
M_1, M_2, M_n whose weights are p_1, p_2, p_n respectively
Let the n_c conditions to be satisfied exactly by the most
probable values, when expressed by equations reduced to
the linear form, be

$$a'V_1 + a''V_2 + \quad -L' = 0$$
$$b'V_1 + b''V_2 + \quad -L'' = 0 \qquad (1)$$

where a', a'',   , b', b'',   ,   , L', L'',   are known
constants

If v_1, v_2, v_n denote the most probable corrections to
the observed values, so that

$$V_1 - M_1 = v_1$$
$$V_2 - M_2 = v_2$$
$$V_n - M_n = v_n$$

$$(2)$$

we have the reduced condition equations

$$a'v_1 + a''v_2 + . \qquad l' \qquad 0$$
$$b'v_1 + b''v_2 + . \qquad l'' \qquad 0$$

or

$$[av] \quad l' \quad 0$$
$$[bv] \quad l'' \quad 0 \qquad\qquad (4)$$

where $l' = L' - [aM], l'' - L'' \quad [b.M], . \quad .,$ and are, there fore, known quantities.

The most probable system of corrections is that which makes

$$[pvv] = \text{a minimum, } \omega \text{ suppose.}$$

The problem is to solve this minimum function when the corrections v are subject to the above n_c conditions.

Direct Solution—Method of Independent Unknowns.

109   It is plain that n_c of the corrections can, by means of the condition equations, be expressed in terms of the remaining $n - n_c$ corrections, and that by substituting these n_c values in the minimum function we should have a reduced minimum function containing $n - n_c$ independent unknowns   This function can be found in the usual way by equating to zero its differential coefficients with respect to each unknown in succession.   The $n - n_c$ resulting equations, taken in connection with the n_c condition equations, determine the n corrections $v_1, v_2, \ldots v_n.$ Thence $[pvv]$ is found

The solution of the $n - n_c$ equations can be carried through by any of the methods of Chapter IV.   The pre cision of the adjusted values, or of any function of them, can also be found as in Chapter IV.

Ex 1 Take that already solved in Ex 4, Art 83

Let v_1, v_2, v_3, v_4, v_5 be the most probable corrections to the measured angles, then the conditions to be satisfied are

$$PSB + v_1 = FSB + v_3 - FSP - v_1$$
$$OSB + v_5 = FSB + v_4 - FSO - v_2$$

Substituting for PSB, FSB, etc , their measured values, the condition equations may be written

$$v_1 - v_1 + v_4 = -0\ 76$$
$$v_2 - v_3 + v_5 = -1\ 66$$

with

$$5v_1^2 + 7v_2^2 + 4v_3^2 + 7v_4^2 + 4v_5^2 = a\ min$$

Substitute for v_4, v_5 in the minimum equation, and

$$5v_1^2 + 7v_2^2 + 4v_3^2 + 7(v_1 - v_3 + 0\ 76)^2 + 4(v_2 - v_3 + 1\ 66)^2 = a\ min$$

Hence, differentiating with respect to v_1, v_2, v_3 as independent variables, we have the normal equations

$$12v_1 \qquad - 7v_3 = -\ 5\ 32$$
$$\qquad 11v_2 - 4v_3 = -\ 6\ 64$$
$$-7v_1 - 4v_2 + 15v_3 = \ 11\ 96$$

whence

$$v_1 = -0''\ 05, \quad v_2 = -0''\ 36, \quad v_3 = +0''\ 68$$

and from the condition equations

$$v_4 = -0''\ 03 \quad v_5 = -0''\ 62$$

These results are the same as those already found on p 149

Ex 2 The angles A, B, C of a spherical triangle are equally well measured , required the adjusted values and their weights

The condition equation to be satisfied is

$$A + B + C = 180 + \varepsilon \qquad (1)$$

where ε is the spherical excess of the triangle

Putting $M_1 + v_1$, $M_2 + v_2$, $M_3 + v_3$ for A, B, C, the condition equation becomes

$$v_1 + v_2 + v_3 = 180 + \varepsilon - [M]$$
$$= l \text{ suppose} \qquad (2)$$

Also

$$v_1^2 + v_2^2 + v_3^2 = a\ min$$

Substitute for v_3 from (2) in the minimum function, and

$$v_1^2 + v_2^2 + (v_1 + v_2 - l)^2 = a\ min$$

Differentiating with respect to the independent variables c_1, c_2, and

$$2c_1 + v_2 = l$$
$$v_1 + 2v_2 \quad l \tag{3}$$

which give

$$v_1 - v_2 = \frac{l}{3}$$

Also from Eq 2,

$$v_1 = \frac{l}{3}$$

Hence *the correction to each angle is one-third of the difference of the theoretical and measured sums of the three angles*

To find the weight of the adjusted value of an angle, as l
The function is

$$dF = c_1$$

Hence, following the method of Art 101 (b),

$$u_F = \frac{1}{\text{wt}} \quad |GQ|$$

where $G_1 = 1$, and Q_1, Q_2 are found from

$$2Q_1 + Q_2 \quad 1$$
$$Q_1 + 2Q_2 \quad 0$$

that is, weight of $A = \frac{2}{3}$ if weight of measured value is unity

Check Weight of direct measure of l 1
Wt of indirect meas $(-180 + l \quad B \quad C)$ of l $\frac{1}{2}$

Weight of mean $1\frac{1}{2}$

as already found

Ex 3 To find the weight of a side, a, in a triangle all of whose angles have been equally well measured, the base, b, being free from error.

Here $F = a - b \dfrac{\sin A}{\sin B}$

$$dF = a \sin 1'' (\cot A \ v_1 - \cot B \ v_2)$$

The weight is found from

$$u_F = a \sin 1'' \cot A \ Q_1 - a \sin 1'' \cot B \ Q_2$$

where Q_1, Q_2 satisfy the equations (Art 101)

$$2Q_1 + Q_2 = a \sin 1'' \cot A$$
$$Q_1 + 2Q_2 = -a \sin 1'' \cot B$$

Hence

$$u_F = \frac{2}{3} a^2 \sin^2 1'' (\cot^2 A + \cot^2 B + \cot A \cot B)$$

Ex 4 The measured values of the angles of a triangle have the same weight Show that if the corrections to the angles are expressed in terms of the corrections to the log sines of the angles, and the corrections to these log sines found by treating them as observed quantities, the same results will be obtained as in Ex 2

For example take 50°, 60°, 70° 00' 30″

Indirect Solution—Method of Correlates

110 If the unknowns in the condition equations are much entangled the direct solution would be very laborious It is in general, therefore, advisable, instead of eliminating the n_c unknowns directly, to do so indirectly by means of undetermined multipliers, or correlates, as they are called

If we multiply the condition equations 3, Art 108, in order by the correlates k', k'', , we may write

$$\omega = k' \left([av] - l'\right) + k''([bv] - l'') + \quad + [pvv] = \text{a min} \quad (1)$$

and determine k', k'', accordingly

By differentiation,

$$d\omega = (a'k' + b'k'' + \quad + 2p_1v_1)dv_1$$
$$+ (a''k' + b''k'' + \quad + 2p_2v_2)dv_2 + \quad (2)$$

If we place equal to zero the coefficients of n_c of the differentials dv_1, dv_2, we shall have n_c equations from which to find k', k'', Substitute these n_c values in the expression for $d\omega$, and there will remain $n - n_c$ differentials which are independent of one another In order that the function may satisfy the condition of a minimum, the coefficients of each of these differentials must be equal to zero This gives $n - n_c$ equations, which equations, taken in connection with the n_c condition equations, give the n unknowns v_1, v_2, v_n

The practical solution would, therefore, be Form n equations by placing equal to zero the differential coefficients of the minimum function with respect to each of the quantities v_1, v_2, v_n From these n equations and the n_c condition equations determine the $n + n_c$ unknowns k', k'', , v_1, v_2, , and thence the function $[pvv]$.

In carrying this out the form of the differential equation 2 shows that it would be advantageous to multiply the minimum equation by $-\frac{1}{2}$, and so write (1) in the form

$$k'([av]-l')+k''([bv]-l'')+ \quad -\tfrac{1}{2}[pvv] = \text{a min} \quad (3)$$

Differentiating, we have the n correlate equations

$$
\begin{aligned}
a'k'+b'k''+ \quad &=p_1v_1\\
a''k'+b''k''+ \quad &=p_2v_2
\end{aligned}
\quad (4)
$$

.

Substituting for $v_1, v_2,$ in the condition equations their values derived from these equations, and the normal equations result   They are

$$
\left[\frac{aa}{p}\right]k' + \left[\frac{ab}{p}\right]k'' + \quad =l'
$$

$$
\left[\frac{ab}{p}\right]k' + \left[\frac{bb}{p}\right]k'' + \quad =l'' \quad (5)
$$

Solving, we obtain $k', k'',$, and thence $v_1, v_2,$ from (4), and $V_1, V_2,$ from (2), Art 108
The normal equations may be written

$$
\begin{aligned}
[uaa]k'+[uab]k''+ \quad &=l'\\
[uab]k+[ubb]k''+ \quad &=l''
\end{aligned}
\quad (6)
$$

. .

where $u_1, u_2,$ denote the reciprocals of the weights $p_1, p_2,$ The form of these equations shows that the coefficients $[uaa], [uab],$ may be computed as in Art. 85, the corresponding scheme being

		k'	k''	k'''	
v_1	u_1	a'	b'	c'	
v_2	u_2	a''	b''	c''	

If the elimination of the normal equations is performed by the method of substitution (Art 86), we have, by collecting the first equations of the successive groups,

$$\lfloor uaa \rfloor k' + [uab] k'' + [uac] k''' + \quad = l'$$
$$+ [ubb\ 1] k'' + \lfloor ubc\ 1 \rfloor k''' + \quad = l''\ 1$$
$$+ [ucc\ 2] k''' + \quad = l'''\ 2 \qquad (7)$$

where l', $l''\ 1$, $l'''\ 2$,   correspond to $[al]$, $[bl\ 1]$, $[cl\ 2]$, respectively

These equations being precisely similar in form to Eq 8, Art 86, the elimination gives (see Art 96)

$$k' = \frac{l'}{\lfloor uaa \rfloor} + \frac{l''\ 1}{[ubb\ 1]} R' + \frac{l'''\ 2}{[ucc\ 2]} R'' +$$

$$k'' = \qquad \frac{l''\ 1}{[ubb\ 1]} + \frac{l'''\ 2}{[ucc\ 2]} S'' + \qquad (8)$$

where

$$0 = \frac{\lfloor uab \rfloor}{\lfloor uaa \rfloor} + R'$$

$$0 = \frac{\lfloor uac \rfloor}{\lfloor uaa \rfloor} + \frac{[ubc\ 1]}{[ubb\ 1]} R' + R'' \qquad (9)$$

$$0 = \frac{[ubc\ 1]}{\lfloor ubb\ 1 \rfloor} + S''$$

$$l''\ 1 = l'R' + l''$$
$$l'''\ 2 = l'R'' + l''S'' + l''' \qquad (10)$$

Ex 1 Take that solved in Ex 1, Art 108)
The condition equations are

$$v_1 - c_1 + c_4 \quad 0.76$$
$$v_2 - c_2 + b \quad 1.66 \qquad (1)$$

The correlate equations consequently are

$$k' \qquad 5.1$$
$$k'' \quad 7c_2$$
$$k' \quad k'' \quad 4c_4$$
$$k' \qquad 7.4$$
$$k'' \quad 4c_5 \qquad (2)$$

To form the normal equations we may substitute for c_1, c_2 from (2) in (1), or proceed by means of the tabular form on p 218 We find

$$0.5929k' + 0.25k'' \quad 0.76$$
$$0.25k' + 0.6429k'' \quad 1.66$$

The solution of these equations gives

$$k' \quad 0.230 \qquad k'' \quad 2.49^2$$

whence, from the correlate equations,

$$v_1 = -0''.05, \ v_2 = -0''.36, \ v_3 \quad +0''.68, \ c_4 \quad 0.03, \qquad 0.64$$

Check The results satisfy the condition equations

Ex 2 The angles, A, B, C, of a spherical triangle are measured with their weights, p_1, p_2, p_3, required their adjusted values
The condition equation may be written (see Ex 1, Art 109)

$$v_1 + v_2 + v_3 \quad l$$

with
$$[p c^2] \quad \text{a min}$$

The correlate equations are

$$k \quad p_1 v_1$$
$$k \quad p_2 v_2$$
$$k \quad p_3 v_3$$

and the normal equation

$$[u]k \quad l$$

$$v_1 = \frac{u_1}{[u]} l, \quad v_2 \quad \frac{u_2}{[u]} l, \quad v_3 \quad \frac{u_3}{[u]} l$$

Hence the adjusted values are known
When the weights are equal, then

$$v_1 = v_2 = v_3 = \tfrac{1}{3} l$$

the same results as in Ex 2, Art 109

Note If a condition equation is of the form

$$a_1 v_1 + a_2 v_2 + \quad + a_n v_n = l$$

the weights of the measured values being p_1, p_2, p_n, then, proceeding as in the above, we have

$$v_1 = \frac{u_1 a_1}{[uaa]} l, \quad v_2 = \frac{u_2 a_2}{[uaa]} l,$$

This result is very important and will be often referred to

Ex 3 At U S Coast Survey station Pine Mt the following were the angles observed between the surrounding stations in order of azimuth

Jocelyne–Deepwater	65°	11'	52" 500	weight	3
Deepwater–Deakyne,	66°	24'	15" 553	"	3
Deakyne–Burden,	87°	02'	24" 703	"	3
Burden–Jocelyne,	141°	21'	21" 757	"	1

required their most probable values

The condition to be satisfied is that the sum of the angles should be 360°
Now,

sum of measured values	= 359°	59'	54" 513	
theoretical sum	= 360°	00'	00" 000	
residual error	=		5" 487	

Hence, as in the preceding example,

$$\text{correction to each of first three angles} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 1} \times 5" 487$$

$$= 0" 914$$

$$\text{correction to fourth angle} = \frac{1}{\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 1} \times 5" 487$$

$$= 2" 744$$

Ex 4 Given in a triangle, of which the longest side is 4 miles, the three angles measured with equal care and with the values 1", 5", 179° 59' 58", adjust the triangle

[Angles as small as single seconds occur in practice In the primary triangulation (1877) of the Great Lakes, carried out by the U S Engineers, in the neighborhood of the Chicago base two angles were measured with values 0" 815 and 1" 185 respectively]

Ex 5 "In order to find the content of a piece of ground, I measured with a common circumferentor and chain the bearings and lengths of its several sides But upon casting up the difference of latitude and departure

I discovered that some error had been contracted in taking the dimensions
Now, it is required to compute the area of this enclosure on the *most probable
supposition* of this error

"Let *ABCDE* be a survey accurately protracted according to the
measured lengths and bearings of the sides *AB*, *BC*,   *A* the place of be-
ginning, *E* of ending, *AG* a meridian, *AF*, *FE*
the errors in latitude and departure   Now,
the problem requires us to make such changes
in the positions of the points *B*, *C*,   that
we may remove the errors *AF*, *FE*—in other
words, that *E* may coincide with *A*, and
these changes must be made in the most
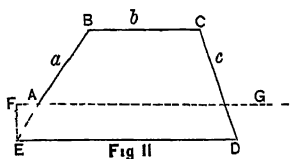
B   *b*   C
a   *c*
F A---------------G - -
/
E   Fig II   D

probable manner   We have, therefore, to fulfil the three following con-
ditions
  "All the changes in departure must remove the error in departure *EF*
  "All the changes in latitude must remove the error in latitude *AF*
  "The probability of these changes must be a maximum "
  Let $a_1, a_2, a_3,$, $\vartheta_1, \vartheta_2, \vartheta_3,$, denote the measured lengths and
bearings of the sides *AB*, *BC*,   , and $x_1, x_2, x_3,$, $y_1, y_2, y_3,$ their
most probable corrections
  Now, since the corrected latitudes must balance, and the corrected de-
partures must also balance, we have the conditions

$$(a_1 + x_1) \cos (\vartheta_1 + y_1) + (a_2 + x_2) \cos (\vartheta_2 + y_2) +   = 0$$
$$(a_1 + x_1) \sin (\vartheta_1 + y_1) + (a_2 + x_2) \sin (\vartheta_2 + y_2) +   = 0$$

or, reducing to the linear form,

$$\cos\vartheta_1\, x_1 - a_1 \sin\vartheta_1\, y_1 + \cos\vartheta_2\, x_2 - a_2 \sin\vartheta_2\, y_2 +   + [a \cos \vartheta] = 0$$
$$\sin\vartheta_1\, x_1 + a_1 \cos\vartheta_1\, y_1 + \sin\vartheta_2\, x_2 + a_2 \cos\vartheta_2\, y_2 +   + [a \sin \vartheta] = 0 \quad (1)$$

with

$$[px^2] + [qy^2] = \text{a minimum},$$

the weights of $x_1, x_2,$, $y_1, y_2,$ being $p_1, p_2,$, $q_1, q_2,$ re-
spectively
  Hence the correlate equations

$$+ \cos\vartheta_1\, k' + \sin\vartheta_1\, k''   = p_1 x_1$$
$$- a_1 \sin\vartheta_1\, k' + a_1 \cos\vartheta_1\, k'' = q_1 y_1 \quad (2)$$

and the normal equations

$$\left\{ \left[\frac{\cos^2 \vartheta}{p} \right] + \left[\frac{a^2 \sin^2 \vartheta}{q} \right] \right\} k' + \left\{ \left[\frac{\sin \vartheta \cos \vartheta}{p} \right] - \left[\frac{a^2 \sin \vartheta \cos \vartheta}{q} \right] \right\} k''$$
$$= - [a \cos \vartheta]$$

$$\left\{ \left[\frac{\sin \vartheta \cos \vartheta}{p} \right] - \left[\frac{a^2 \sin \vartheta \cos \vartheta}{q} \right] \right\} k' + \left\{ \left[\frac{\sin^2 \vartheta}{p} \right] + \left[\frac{a^2 \cos^2 \vartheta}{q} \right] \right\} k''$$
$$= - [a \sin \vartheta]$$

Now if we assume

$$p_1 = \frac{1}{a_1}, \; p_2 = \frac{1}{a_2},$$

$$q_1 = a_1, \; q_2 = a_2,$$

as "this seems best to agree with the imperfections of the common instruments used in surveying," the normal equations reduce to

$$\lambda'[a] \qquad = -[a \cos \vartheta]$$
$$\lambda''[a] = -[a \sin \vartheta]$$

from which λ', λ'' are known

The corrections x_1, x_2,   , $v_1 \; y_2$, are known from (2)
The errors in latitude (see Eq 1) now reduce to

$$\cos\vartheta_1 \, x_1 - a_1 \sin\vartheta_1 \, v_1 = -a_1 \frac{[a \cos \vartheta]}{[a]}$$

$$\cos\vartheta_2 \, x_2 - a_2 \sin\vartheta_2 \, v_2 = -a_2 \frac{[a \cos \vartheta]}{[a]}$$

and the errors in departure to

$$\sin\vartheta_1 \, x_1 + a_1 \cos\vartheta_1 \, y_1 = -a_1 \frac{[a \sin \vartheta]}{[a]}$$

$$\sin\vartheta_2 \, x_2 + a_2 \sin\vartheta_2 \, y_2 = -a_2 \frac{[a \sin \vartheta]}{[a]}$$

Hence Bowditch's *rule for balancing a survey* "Say as the sum of all the distances is to each particular distance, so is the whole error in departure to the correction of the corresponding departure, each correction being so applied as to diminish the whole error in departure   Proceed in same way for the correction in latitude "

This problem was proposed as a prize question by Robert Patterson, of Philadelphia, in vol 1 No 3 of the *Analyst or Mathematical Museum*, edited by Dr Adrain, of Reading, Pa , and published in 1807 In vol 1 No 4 are two solutions—one by Bowditch, to whom the prize was awarded, and the other by Dr Adrain   Adrain's mode of solution is nearly the same as by the ordinary Gaussian method He employs undetermined multipliers or correlates, exactly as Gauss subsequently did   To Adrain, therefore, is due not only the first derivation of the exponential law of error, but its first application to geodetic work   See Appendix I

To Find the Precision of the Adjusted Values, or of any Func tion of them

111   The method of proceeding is the same as in Art 101

The first step is to find μ, the m s e. of a single observa tion, and next the weight, p_F, of the function, whence the m s e of the function is given by

$$\mu \cdot u_F$$

u_F being the reciprocal of the weight

(a) To find μ

In Art 99 it was shown that in a system of observation equations the m s e μ of an observation of the unit of weight is found from

$$\mu = \sqrt{\frac{|pvv|}{n - n_i}}$$

where $|pvv|$ is the sum of the weighted squares of the residuals v, n is the number of observation equations, and n_i the number of independent unknowns

Hence in a system of condition equations, n being the number of observed quantities and n_i the number of con ditions, the number of independent unknowns is $n - n_i$ and

$$\mu = \sqrt{\frac{|pvv|}{n - (n - n_i)}}$$

$$\mu = \sqrt{\frac{|pvv|}{n_i}} \qquad (1)$$

Luroth's formula (Art 99) may be used as a check on the value of μ.

Checks of $|pvv|$ —When the number of residuals is large, in order to guard against mistakes $|pvv|$ should be com puted in at least two different ways   The following check methods will be found useful

(α) The correlate equations 4, Art 110, may be written

$$\sqrt{p_1}\, v_1 = \sqrt{u_1}\, a'\, k' + \sqrt{u_1}\, b'\, k'' +$$
$$\sqrt{p_2}\, v_2 = \sqrt{u_2}\, a''k' + \sqrt{u_2}\, b''k'' +$$

Square and add, and

$$|\,pvv\,| = \lfloor uaa \rfloor k'k' + 2[uab]k'\,k'' + 2[uac]k'k''' \,+$$
$$+\; [ubb]k''k'' + 2[ubc]k''k''' \,+$$
$$+\; [ucc]k'''k''' + \qquad (2)$$
$$+\qquad .$$

$$= |\,kl\,| \text{ from (6), Art } 110$$

(β) $|\,pvv\,| = |\,kl\,|$

$$= l'k' + l''k'' +$$

$$= l'\left(\frac{l'}{\lceil uaa \rceil} + \frac{l''\ \mathrm{I}}{\lceil ubb\ \mathrm{I}\rceil}R' + \frac{l'''\ 2}{\lceil ucc\ 2\rceil}R'' + \qquad \right)$$

$$+\; l''\left(\frac{l''\ \mathrm{I}}{\lceil ubb\ \mathrm{I}\rceil} + \frac{l'''\ 2}{\lceil ucc\ 2\rceil}S'' + \qquad \right)$$

$$+$$

$$= \frac{(l')^2}{\lceil uaa\rceil} + \frac{(l''\ \mathrm{I})^2}{\lceil ubb\ \mathrm{I}\rceil} + \frac{(l'''\ 2)^2}{\lceil ucc\ 2\rceil} + \qquad (3)$$

by addition attending to Eq 10, Art 110

This expression is very readily computed from the solution of the correlate normal equations, as shown in Ex 2 following Compare the computation of $[vv]$ from the scheme in Art 100.

The sum $|\,pvv\,|$ can in general be computed more rapidly by these methods than by the direct process of summing the weighted squares of the residuals

Ex 1 The three angles of a triangle are measured with the weights p_1, p_2, p_3, required the mean-square error of a single observation

Using the values of v_1, v_2, v_3 found in Ex 2, Art 110, we have

$$[pvv] = \frac{u_1 l^2}{[u]^2} + \frac{u_2 l^2}{[u]^2} + \frac{u_3 l^2}{[u]^2}$$
$$= \frac{l^2}{[u]}$$

Hence

$$\mu = \frac{l}{\sqrt{[u]}} \quad \text{since } n_t = 1$$

Check (1) $[p\eta\eta'] = [kl]$

$$= \frac{l}{[u]}l$$

as before

Check (2) $[p\eta\eta'] = \frac{(l)^2}{[u]}$ directly from Eq 3, since $[uau] = 1$

F_3 2 To find the m s e of a single observation in Ex 1, Art 110
The first step is to find the value of $[p\omega]$ Three methods are given

(1)

p	c	pcc
5	0 05	0125
7	- 0 36	9072
4	+ 0 68	1 8496
7	0 03	0063
4	0 62	1 5376
		4 3132 $[p\omega]$

(2)

k	l	kl
— 0 230	0 76	0 1748
— 2 493	1 66	4 1384
		4 3132 $[p\omega]$

(3) From the solution of the correlate normal equations

k'	k''		
+ 0 5929	+ 0 2500	0 76	l'
+ 0 2500	+ 0 6420	1 66	l
+ 1	+ 0 4217	1 2818	l' $[u\,aa]$
	+ 0 5375	1 3395	l'' 1
	+ 1	2 492	$l''.1$ $[ubb.1]$

$$[pvv] = 0\ 76 \times 1\ 2819 + 1\ 3395 \times 2\ 492$$
$$= 4\ 3136$$

Hence, the number of conditions being two,

$$\mu = \sqrt{\frac{4\ 31}{2}} = 1''\ 47$$

(b) To find u_F

Let the function whose weight is to be found be

$$F = f(V_1, V_2, \qquad V_n) \tag{4}$$

and let it be conditioned by the n_c equations

$$f_1(V_1, V_2, \qquad V_n) = 0$$
$$f_2(V_1, V_2, \qquad V_n) = 0 \tag{5}$$

Expressing F in terms of the observed values, $M_1, M_2, \qquad M_n$, which are independent of one another, and reducing to the linear form, we have

$$dF = \frac{\partial F}{\partial M_1} v_1 + \frac{\partial F}{\partial M_2} v_2 + \tag{6}$$

Hence as in Art 101,

$$u_F = u_1 \left(\frac{\partial F}{\partial M_1} \right)^2 + u_2 \left(\frac{\partial F}{\partial M_2} \right)^2 + \tag{7}$$

where u_1, u_2 are the reciprocals of the weights of the observed values

As it usually requires a long elimination to express F in terms of $M_1, M_2, \qquad M_n$ directly, it is better to compute $\dfrac{\partial F}{\partial M_1}, \dfrac{\partial F}{\partial M_2}$, from the forms

$$\frac{\partial F}{\partial M_1} = \frac{\partial F}{\partial V_1} \frac{\partial V_1}{\partial M_1} + \frac{\partial F}{\partial V_2} \frac{\partial V_2}{\partial M_1} +$$

$$\frac{\partial F}{\partial M_2} = \frac{\partial F}{\partial V_1} \frac{\partial V_1}{\partial M_2} + \frac{\partial F}{\partial V_2} \frac{\partial V_2}{\partial M_2} + \tag{8}$$

.   .   .   .

30

Ex 3 To find the m s e of a side, a, in a triangle whose angles have been measured with the weights p_1, p_2, p_3, the base, c, being free from error

The function equation is

$$b = a\; b \frac{\sin I}{\sin B}$$

and the condition equation

$$I + B + C = 180 + e$$

Hence from Ex 2, Art 110, expressing I, b in terms of the observed values

$$A = M_1 + \frac{u_1}{[u]} \{180 + e - (M_1 + M_2 + M_3)\}$$

$$B = M_2 + \frac{u_2}{[u]} \{180 + e - (M_1 + M_2 + M_3)\}$$

Now,

$$db = \left(\frac{\partial b}{\partial I}\frac{\partial I}{\partial M_1} + \frac{\partial b}{\partial B}\frac{\partial B}{\partial M_1}\right) +$$

$$+ \left(\frac{\partial b}{\partial I}\frac{\partial I}{\partial M_2} + \frac{\partial I}{\partial B}\frac{\partial B}{\partial M_2}\right) + \left(\frac{\partial I}{\partial I}\frac{\partial I}{\partial M} + \frac{\partial I}{\partial B}\frac{\partial B}{\partial M}\right)$$

$$a \sin I'' \left(\left\{ \left(1 - \frac{u_1}{[u]}\right) \cot I + \frac{u_2}{[u]} \cot B \right\} +$$

$$+ \left\{ - \frac{u_1}{[u]} \cot I + \left(1 - \frac{u_2}{[u]}\right) \cot B \right\} + \left\{ - \frac{u_1}{[u]} \cot A - \frac{u}{[u]} \cot B \right\} \right)$$

Therefore

$$u_b = a^2 \sin^2 I'' \left(\left\{ \left(1 - \frac{u_1}{[u]}\right) \cot I + \frac{u_2}{[u]} \cot B \right\}' u_1$$

$$+ \left\{ \frac{u_1}{[u]} \cot I + \left(1 - \frac{u_1}{[u]}\right) \cot B \right\} u_2 + \left\{ \frac{u_1}{[u]} \cot I - \frac{u_1}{[u]} \cot B \right\} u \right)$$

$$a^2 \sin^2 I'' \left\{ \left(u_1 - \frac{u_1''}{[u]}\right) \cot^2 I + \left(u_2 - \frac{u_1'}{[u]}\right) \cot^2 B + \frac{2u_1 u}{[u]} \cot I \cot B \right\}$$

and

$$u_b = \mu + u_b$$

where μ is the m s e of a single observation

If the weights p_1, p_2, p_3 are each equal to unity, this reduces to

$$u_b^2 = \tfrac{2}{3} a^2 \sin^2 I'' \mu^2 (\cot^2 A + \cot^2 B + \cot I \cot B)$$

and if the triangle is equilateral,

$$\mu_b^2 = \tfrac{2}{3} a^2 \sin^2 I'' \mu^2$$

Also, if the base, instead of being considered exact, had the m s e u_c, the expressions for μ_b^2 would be increased by $\frac{u''}{b^2} u_c^2$ and u_c represents

It is, however, usually much more convenient in practice to use the method of correlates

Let the function, reduced to the linear form, be written

$$dF = f + cz + fz + \ldots \qquad (9)$$

This is conditioned by the n equations, also in the linear form,

$$
\begin{aligned}
az + a z + \ldots && l &= 0 \\
a z + l z + \ldots && l' &= 0 && (10)
\end{aligned}
$$

with

$$[p z^2] = \text{a minimum}$$

Referring to the principle of Art. 110, we see that by using correlates $k, k', \ldots$, and determining them properly, we can express the function in terms of the quantities $z, z', \ldots z_n$ as if independent, that is,

$$
dF = \{ f - z + \ldots \\
+ z f' - a k - b'k \ldots \}z + \ldots \qquad (11)
$$

and, therefore,

$$
u_k = \{ f - a k - b k \ldots \}^2 u_1 \\
+ \{ f - a'k - b'k \ldots \}^2 u_1 \} \qquad (12)
$$

It remains to determine $k', k', \ldots$. Now, when the most probable values of the corrections $z_1, z_2, \ldots z_n$ are substituted in the value of the function dF, this function must have its most probable value, and, therefore, its maximum weight. We may, therefore, determine the correlates k from the condition that the weight of dF is a maximum; that is, that u_k is a minimum. Differentiate, then, u_k with respect to $k', k', \ldots$ as independent variables, and we have the equations

$$
[aa]k' + [ab]k'' + \ldots \quad [af] \\
[ab]k' + [bb]k'' + \ldots \quad [bf] \qquad (13)
$$

from which $k', k', \ldots$ are found.

These equations being precisely of the form of ordinary normal equations, it follows, as in (c) and (d), Art. 100, that

$$u_k = \lfloor uff \rfloor - \lfloor uaf \rfloor k' - \lfloor ubf \rfloor k'' \qquad (14)$$

or

$$u_k = \lfloor uff \rfloor - \frac{\lfloor ua f \rfloor^2}{\lfloor uaa \rfloor} - \frac{\lfloor ubf\ 1 \rfloor^2}{\lfloor ubb\ 1 \rfloor} \qquad . \qquad (15)$$

The form of the last expression for u_k shows that it may be found by means of the following scheme, in which $\lfloor uaf \rfloor$, $\lfloor ubf \rfloor$, are added as an extra column in the solution of the correlate normal equations (13), in the manner shown in Art 100   For three correlates the scheme would be

k'	k''	k'''	
- ---			
$[uaa]$	$\lfloor uab \rfloor$	$\lfloor uac \rfloor$	$\lfloor uaf \rfloor$
	$[ubb]$	$\lfloor ubc \rfloor$	$\lfloor ubf \rfloor$
		$\lfloor ucc \rfloor$	$\lfloor ucf \rfloor$
			$\lfloor uff \rfloor$
	$\lfloor ubb\ 1 \rfloor$	$\lfloor ubc\ 1 \rfloor$	$\lfloor ubf\ 1 \rfloor$
		$\lfloor ucc\ 1 \rfloor$	$\lfloor ucf\ 1 \rfloor$
			$\lfloor uff.1 \rfloor$
		$\lfloor ucc\ 2 \rfloor$	$\lfloor ucf.2 \rfloor$
			$\lfloor uff.2 \rfloor$
			$\lfloor uff\ 3 \rfloor$
			u_k

Ex 4 To find the weight of the angle PSB in Ex 1, Art 109
Here
$$dF = -v_1 + v_1$$
$$f_1 = -1, \quad f_2 = 0, \quad f_3 = +1$$

From the condition equations

$$a = +1 \qquad b'' = +1$$
$$a'' = -1 \qquad b''' = -1$$
$$a'''' = +1 \qquad b''''' = +1$$

$$[uaf] = \tfrac{1}{6} \times -1 + 1 \times -1 = -0.45$$
$$[ubf] = -\tfrac{1}{4}$$
$$[ucf] = +0.45$$

The correlate normal equations with the extra column for finding u_F

k	k''	l		
+ 0 5929	+ 0 2500	− 0 7600	− 0 4500 = $[uaf]$	
+ 0 2500	+ 0 6429	− 1 6600	+ 0 2500 = $[ubf]$	
				+ 0 4500 = $[uff]$
+ 1	+ 0 4217	− 1 2818	− 0 7590	
	+ 0 5375	− 1 3395	− 0 0602 = $[ubf\,1]$	
				+ 0 3416
				+ 0 1084 = $[uff\,1]$
	+ 1	− 2 492	− 0 1120	
				+ 0 0067
				+ 0 1017 = $[uff\,2]$
				= u_F

Also
$$\mu_F = \mu \sqrt{u_F}$$
$$= 1.47 \sqrt{0.1017} \text{ from Ex 2}$$
$$= 0''.47$$

as before

Ex 5 To find the weight and m s e of the adjusted value of an angle of a triangle when all three angles are measured, their weights being p_1, p_2, p_3 respectively
The function is
$$dF = v_1$$
and the condition equation
$$v_1 + v_2 + v_3 = l$$

Hence from (15)

$$u_f \quad u_1 \quad \frac{u_3}{|u|}$$

$$\frac{u_1(u + u)}{|u|}$$

Also
$$u_p \quad u + u_f$$

$$\frac{l}{4|u|} \left\{ \frac{u_1(u_1 + u_3)}{|u|} \right. \quad \text{(See Ex. 1)}$$

$$\frac{l}{|u|} \left\{ u_1(u + u_3) \right.$$

The weight of an angle before adjustment is to the weight after adjustment

$$\frac{1}{u_1} \quad \frac{|u|}{u_1(u_1 + u_3)}$$

or
$$u_3 + u_1 \quad |u|$$

If $p_1 \quad p_2 \quad p_3 \quad 1$, the weights are as   The result is independent of
the magnitude of the angle   It therefore applies to any problem in which
the condition to be satisfied is that the sum of two quantities shall be equal to
a third, or in which the sum of all three is equal to a constant   For other
solutions see Ex. 2, Art. 109.

Ex. 6 If n angles measured at a station close the horizon, find the weight
of the adjusted value of any one of them

[The solution is exactly as in the preceding example

The weight of V_1, for instance, is found from

$$u_{I_1} \quad \frac{u_1 \}|u| \quad n_1 \{}{|u|}$$

If the weights $p_1, p_2,$ are all equal to one another, the weight of an angle
after adjustment is to its weight before adjustment is

$$n \quad n \quad 1 \quad]$$

Ex. 7 Show that the weight of the sum of the adjusted angles of a triangle
is infinite

[Sum   180 + t, a fixed quantity,

∴ m. s. e.   0, and weight   ∞

or otherwise

$$a_1 + a_2 + a_3 \quad l$$
$$a' \quad a' \quad a'' \quad 1$$

$$u_p \quad 3 \quad \frac{1^0}{3} \quad 0 \quad]$$

Ex 8 In the "longitude triangle" Brest, Greenwich, Paris, as determined by the U S Coast Survey in 1872, the observed values were

	m	s		
Brest–Greenwich,	17	57 154	weight	10
Greenwich–Paris,	9	21 120	"	7
Brest–Paris,	27	18 190	"	9

Show that the most probable values are

	m	s		
	17	57 130	weight	14
	9	21 086	"	12
	27	18 216	"	13

Ex 9 To find the weight of a side in a chain of triangles, all of the angles of each triangle having been equally well measured and the base being free from error

Let b be the measured value of the base, and let $a_1, a_2, \quad a_n$ be the sides of continuation in order as computed from b, a_n being the side whose weight is required

Fig 12
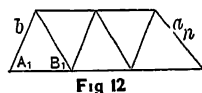
If $A_1, B_1, A_2, B_2,$ are the measured values of the angles used in computing a_n from b, the angles $A_1, A_2,$ being opposite to the sides of continuation, then

$$\frac{a_1}{b} = \frac{\sin A_1}{\sin B_1}, \quad \frac{a_2}{a_1} = \frac{\sin A_2}{\sin B_2}, \quad , \frac{a_n}{a_{n-1}} = \frac{\sin A_n}{\sin B_n}$$

Hence by multiplying these expressions together,

$$a_n = b \frac{\sin A_1}{\sin B_1} \frac{\sin A_2}{\sin B_2} \quad \frac{\sin A_n}{\sin B_n} \tag{1}$$

We may now proceed in two ways

(a) Differentiating directly,

$$da_n = a_n \sin 1'' \left[\cot A \, (A) - \cot B \, (B) \right]$$

where (A), (B), denote the corrections to A, B,

[In a chain of triangles it is convenient to use the notation (A), (B) for $v_1, v_2,$, the parentheses indicating corrections]

The condition equations, from the closure of the triangles, are

$$(A_1) + (B_1) + (C_1) = l'$$
$$(A_2) + (B_2) + (C_2) = l'' \tag{2}$$

Substituting in Eq, 15,

$$u_{a_n} = \tfrac{2}{3} a_n^2 \sin^2 1'' \left[\cot^2 A + \cot^2 B + \cot A \cot B \right] \tag{3}$$

the result required

If the triangles are equilateral this reduces to

$$u_{a_n} = \tfrac{2}{3} n b^2 \sin^2 1'' \tag{4}$$

Hence in a chain of equilateral triangles the weights of the sides decrease as we proceed from the base, b, through the successive triangles, inversely as the number of triangles passed over, that is, are as the fractions

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4},$$

(b) Taking logs of both members of Eq 1 and differentiating,

$$d \log a_n = \frac{d}{d I_1} \log \sin A_1 (A_1) - \frac{d}{d B_1} \log \sin B_1 (B_1) +$$

$$= [\delta_A(A) - \delta_B(B)] \tag{5}$$

or expanding the first member,

$$\delta_{a_n} da_n = [\delta_A(A) - \delta_B(B)] \tag{6}$$

where δ_a is the tabular difference for one unit for the number a_n, and δ_A, δ_b are the logarithmic differences corresponding to $1''$ for the angles A, B in a table of log sines   (See Art 7)

Hence attending to the condition equations 2, we have from (5) for Eq 5,

$$u_{\log a_n} = \tfrac{2}{3}[\delta_A{}^2 + \delta_A \delta_B + \delta_b']$$

and for Eq 6,

$$u_{a_n} = \tfrac{2}{3} \frac{I}{\delta_{a_n}}[\delta_A' + \delta_A \delta_b + \delta_b']$$

as giving the weight of the logarithm of the side and the weight of the side respectively

Of the two forms (a) and (b), the logarithmic is in general the most convenient in practice

Ex 10   From a base $AB (= b)$ proceeds a chain of equilateral triangles, all of the angles being equally well measured, and the sides BC, CD, being in the same straight line   Find the m s e of the line BN, which is n times the base
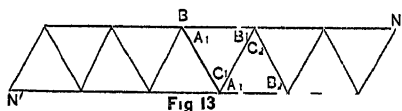
Fig 13

Take first the simple case of $n = 2$

$$F = BN = b \frac{\sin C_1}{\sin B_1} + b \frac{\sin A_1 \sin A_2 \sin C_2}{\sin B_1 \sin B_2 \sin B_2}$$

$$dF = \{ \cot A_1 (A_1) - 2 \cot B_1 (B_1) + \cot C_1 (C_1)$$
$$+ \cot A_2 (A_2) - \cot B_2 (B_2)$$
$$- \cot B_2 (B_2) + \cot C_2 (C_2) \} \, b \sin 1''$$

Also, we have the condition equations

$$(A_1) + (B_1) + (C_1) = l'$$
$$(A_2) + (B_2) + (C_2) = l''$$
$$(A_3) + (B_3) + (C_3) = l'''$$

Hence

$$\begin{bmatrix} \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot \end{bmatrix}$$

Substituting, $\ldots$

and therefore

where $\ldots$ the $\ldots$ of an observed angle

Generally,

and

If the chain proceeds in the opposite direction until $IX = BX$, then since $a_{IX} = a_X$, and $XX = n$ approximately, we have

If XX contains the base (putting $n = \dfrac{n}{2}$)

Hence it follows that in a chain of equilateral triangles where one base only is measured, it is better to place the base at the centre of the chain rather than at either end.

$\ldots$ If a chain of equilateral triangles proceeds from the base IB, as the chain in IX etc, but in the opposite direction, show that the measure of BX, which is n times the base, is

and if the chain proceeds as in the opposite direction until $IX = BX$, then if XX be taken n times the base,

Ex 12 If a chain of equilateral triangles proceeds from the base b, which is in the same straight line as the derived side BN, show that the m s e of BN, which is at ... of the base b_1 is
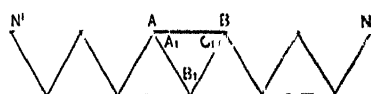
Fig 14

$$\frac{\mu b}{4}\sin 1'' + \{m \cdots \} n_0$$

$$\left[l' = b + b\,\frac{\sin A_1 \sin A_2 \sin C_1}{\sin B_1 \sin B_2 \sin B_3} + b\,\frac{\sin I_1 \sin I_2 \sin I_3 \sin I_4 \sin}{\sin B_1 \sin B_2 \sin I_3 \sin I_4 \sin B_4 \sin t} \right]$$

If also a chain of equilateral triangles proceeds from l' in the opposite direction to N', then if NN' is n times the base, show that the m s e of NN' is

$$\mu NN' \sin 1'' \sqrt{\frac{n^2 + 3m}{9n}} + \cdots$$

Hence show that in computing a line NN', equal to n time the base b, through a chain of equilateral triangles, the least loss of precision with the form of Fig 12

Ex 13 To find the m s e of the altitude of a triangle, the base being supposed free from error, and the reciprocal weights of the angles being u_1, u_2, u_3 respectively

The function is

$$F = b\,\frac{\sin A \sin C}{\sin B}$$

$$dF = F\sin 1''\{\cot A\,(dA) - \cot B\,(dB) + \cot C\,(dC)\}$$

Also the condition

$$(A) + (B) + (C) = l$$

Substituting in (15)

$$u_F = F^2\{u_1\cot^2 A + u_2\cot^2 B + u_3\cot^2 C$$

$$- \frac{(u_1\cot A + u_2\cot B + u_3\cot C)^2}{[u]}\}\sin^2 1''$$

If $A = C$, and $u_1 = u_2 = u_3 = \dfrac{1}{r}$

then

$$u_F = \frac{2}{3}\frac{F^2}{r}\sin^2 1''\cosec^2 B$$

and

$$\mu_F = \frac{\mu b \sin 1''}{\surd 2\,r}\cosec^2 \frac{B}{2}$$

where μ is the m s e of the angle corresponding to the unit of weight

Ex 14 If two similar isosceles triangles on opposite sides of the base AC are measured independently, thus forming a rhombus (vertices B, B'), then, taking the weight of each angle unity,

$$\mu_{BB'} = \sqrt{2}\ \frac{\mu b \sin 1''}{\sqrt{24}}\ \csc^2 \frac{B}{2}$$

and if BB' is n times the base b, then, since $\cot \frac{B}{2} = n$,

$$\mu_{bB'} = \frac{\mu BB' \sin 1''}{2\sqrt{3}}\left(n + \frac{1}{n}\right)$$

Caution —If we solved for the rhombus directly it would not do to take

$$BB' = b \cot \frac{B}{2}$$

and then form $\mu_{BB'}$ The result would be $\sqrt{2}$ times too great For as the triangles are measured independently, each half of BB' must be considered separately, so that we must use the form

$$BB' = \frac{b}{2}\left(\cot \frac{B}{2} + \cot \frac{B'}{2}\right)$$

with the condition equations

$$(A) + (B) + (C) = l_1$$
$$(A') + (B') + (C') = l_2$$

corresponding to the angles of the two triangles

Ex 15 If on a base, b, is diagonal two similar isosceles triangles are described, forming a rhombus, and on the other diagonal of this rhombus two triangles similar to the former are described, forming a second rhombus, and so on m times, required the m s e of the 1st diagonal, all of the angles being equally well measured

For the mth diagonal d

$$F = d = \frac{b}{2^m}\left(\cot \frac{B_1}{2} + \cot \frac{B_2}{2}\right)\left(\cot \frac{B_3}{2} + \cot \frac{B_4}{2}\right)$$
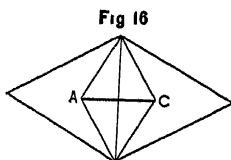
where B_1, B_4, are the vertical angles in order

Now, as the triangles are all similar,

$$B_1 = B_2 = B_3 = = B_{2m} = B \text{ suppose}$$

Hence

$$f' = f'' = = \frac{b}{4} \cot^{m-1} \frac{B}{2} \csc^2 \frac{B}{2} \sin 1''$$
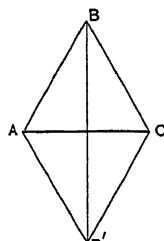
Fig 15

Fig 16

and

$$u_d = 2m \left\{ \left(\frac{b^2}{16} \cot m + \frac{b}{2} \csc^2 \frac{b}{2} \right) + \frac{1}{4} \left(\cot \cdots \right) \right\}$$

$$\frac{mb^2}{12} \cot m + \frac{b}{2} \csc^2 \frac{b}{2} \sin 1$$

But

$$d = c \cot^m \frac{b}{\cdot}$$

$$u_d = \frac{m}{4} d \, c \, c \, c \, b \, m \, 1$$

If d is n times the base,

$$u_d = ud \left\{ \frac{cm}{4} + \frac{1+n}{n^2} \right\} \sin 1$$

For further development of this subject consult Helmert *rationelle Vermessungen* Leipzig, 1865

Solution in Two Groups

112 In geodetic work it often happens that the observed quantities are subject to a simple set of conditions which may be readily solved as observation equations by the method of independent unknowns, and are subjected to other conditions which are best solved by the method of correlates. The equations are thus divided into two groups for solution, and the complete solution, therefore, consists of two parts. The observation equations forming the first group are solved by themselves and give approximations to the final values of the unknowns. The corrections to these approximate values due to the second group are next found by solving this second group by the method of correlates.

The merit of the method consists in utilizing the work expended in the solution of the first group in determining the additional corrections due to the second group. The

solution is rigorous, and, being broken into two parts, is more easily managed than if all of the equations had been solved simultaneously

Let the first group of equations be the observation equations, n in number and containing n_u unknowns ($n > n_u$),

$$a_1x + b_1y + \quad - l_1 = v_1 \quad \text{weight } p_1$$
$$a_2x + b_2y + \quad - l = v_. \quad \text{``} \quad p \quad (1)$$

and the second group the condition equations, n_c in number, involving the same unknowns ($n_c < n_u$),

$$a'x + a''y + \quad - l' = 0$$
$$b'x + b''y + \quad - l'' = 0 \quad (2)$$

The most probable values of the unknowns $x, y,$ are those which are given by the relation

$$\lfloor pvv' \rfloor = \text{a minimum} \quad (3)$$

It is required to find them

The value of an unknown is found in two parts, the first, $(1), (y),$, arising from the observation equations, and the second, $(1), (2),$, arising from the condition equations, thus

$$x = (1) + (1)$$
$$y = (y) + (2) \quad (4)$$

       .   .

Now, overlooking for the present the condition equations and taking the observation equations only, $(x), (y),$ would be found by solving these equations in the usual way   We have, therefore, reducing all to weight unity for convenience in writing, the normal equations

$$\lceil aa \rceil(1) + \lfloor ab \rfloor(y) + \quad = [al]$$
$$\lfloor ab \rfloor(1) + \lfloor bb \rfloor(y) + \quad = [bl] \quad (5)$$

       .   .   .   .   .   .

The solution of these equations gives (see Art. 97)

$$(x) = |aa||al| + |a_\beta||bl| +$$
$$(y) = |a_\beta||al| + |\beta\beta||bl| + \qquad (6)$$

Hence (x), (y),   are known

To find the condition corrections (1), (2),   eliminate v_1, v_2, v_n by substituting in the minimum equation, which then becomes

$$|aa|xx + 2|ab|xy + \quad . \quad 2|al|1$$
$$+ |bb|yy + \qquad 2|bl|1 \qquad (7)$$
$$. \quad .$$
$$+ |ll| \quad a\ min.$$

This equation is conditioned by equations 2   Thus the solution is reduced to that already carried out in Art. 116.

Calling I, II,   the correlates of equations ·, we have the correlate equations

$$\lceil aa|x + |ab|y + \quad - |al| \quad a'I + b'II + \ldots$$
$$\lfloor ab|x + |bb|y + \quad - |bl| \quad a''I + b''II +$$
$$. \qquad .$$

These equations, taken with (4) and (5), give the relations

$$[aa](1) + [ab](2) + \quad -a'I + b'II + \qquad |1| \ suppose$$
$$[ab](1) + [bb](2) + \quad -a''I + b''II + \qquad |2| \quad `` \quad (8)$$
$$. \qquad .$$

which being of the same form as (5), then solution gives

$$(1) = [aa]|\overline{1}| + |a_\beta||2| + \quad . \quad .$$
$$(2) = [a_\beta]|1| + |\beta\beta||2| + \qquad . \qquad (9)$$

or substituting for $|\overline{1}|$, $|2|$,   their values from (8),

$$(1) = A'I + B'II + C'III + .$$
$$(2) = A''I + B''II + C''III + \qquad (10)$$
$$. \qquad . \qquad .$$

where

$$\text{A}' \quad |aa|a' \;|\text{-}\;|a_i\beta|a'' \;|\text{-}$$
$$\text{B}' \quad |aa|b' \;|\; |a_i\beta|b'' \;|\text{-} \tag{11}$$

and are known quantities

We have, therefore, expressed the corrections (1), (2), in terms of the unknown correlates $I, II,$ It remains now to find these correlates

Substituting for $x, y,$ their values from (4) in the condition equations, and

$$a'(1) \;|\; a''(2) +\qquad = l_0'$$
$$b'(1) \;|\; b''(2) \;|\qquad - l_0'' \tag{12}$$

where

$$l_0' \quad l' \quad a'(1) \quad a''(y)$$
$$l_0'' \quad l'' \quad b'(1) - b''(y) - \tag{13}$$

and are, therefore, known quantities, since (1), (y),   are known

Substitute the values of (1), (2), , from (10) in (12), and we have the correlate normal equations

$$[a\text{A}] \;I\; |\; [a\text{B}] \;II\; |\qquad l_0'$$
$$[a\text{B}] \;I\; |\; [b\text{B}] \;II\; |\qquad l_0'' \tag{14}$$

where

$$[a\text{A}] \quad |aa|a'a' \;|\; |a_i\beta|a'a'' \;|$$
$$|\; |a_i\beta|a'a'' \;|\; |\beta_i\beta|a''a'' \;| \tag{15}$$

etc.   etc

The solution of these equations gives the correlates $I, II,$.   Hence the corrections (1), (2),   are known. Also, since (1), (y), . have been found from (6), the total corrections $x, y,$. are known.

113 In carrying the preceding solution into practice the following order of procedure will be found convenient

(a) The formation and solution of the observation equations (1)

The partially adjusted resulting values (1), (y),   are now to be used

(b) The formation of the condition equations (12)

$$a'(1) + a''(2) + \quad = l_0'$$
$$b'(1) + b''(2) + \quad = l_0''$$

(c) The formation of the weight equations (9)   They are at once written down from the general solution of the observation equations in (a), and are

$$(1) = [aa]\,\boxed{1} + [\alpha\beta]\,\boxed{2} +$$
$$(2) = [\alpha\beta]\,\boxed{1} + [\beta\beta]\,\boxed{2} + \quad \cdot$$

(d) The formation of the correlate equations (8)

$$\boxed{1} = a'\,I + b'\,II +$$
$$\boxed{2} = a''\,I + b''\,II +$$
$$\cdot \qquad \cdot$$

(e) The expression of the corrections in terms of the correlates by substituting from (d) in (c)

$$(1) = \text{A}'\,I + \text{B}'\,II +$$
$$(2) = \text{A}''\,I + \text{B}''\,II +$$

(f) The formation of the normal equations by substituting from (e) in (b)   They are,

$$[a\text{A}]\,I + [a\text{B}]\,II + \quad = l_0'$$
$$[a\text{B}]\,I + [b\text{B}]\,II + \quad = l_0''$$
$$\cdot$$

(g) The determination of the corrections by substituting the values of the correlates in (e)

114 **To Find the Precision of the Adjusted Values or of any Function of them.**

(a) First find μ, the m s e of an observation of weight unity

We have (Art 111)

$$\mu^2 = \frac{[vv]}{\text{number of conditions}}$$

$$= \frac{[vv]}{(n - n_u) + n_c}$$

since $n - n_u$ is the number of conditions in the observation equations, and n_c the number in the condition equations

To find $[vv]$ From the first observation equation

$$
\begin{aligned}
v_1 &= a_1 x + b_1 y + \qquad - l_1 \\
&= a_1(x) + b_1(y) + \qquad - l_1 + a_1(1) + b_1(2) + \\
&= v_1^0 + a_1(1) + b_1(2) +
\end{aligned}
$$

Similarly

$$v_2 = v_2^0 + a_2(1) + b_2(2) +$$

where

$$
\begin{aligned}
v_1^0 &= a_1(x) + b_1(y) + \qquad - l_1 \\
v_2^0 &= a_2(x) + b_2(y) + \qquad - l_2
\end{aligned}
$$

that is, $v_1^0, v_2^0,$ are the residuals arising from taking the observation equations only

Attending to Eq 5, p 239, it follows evidently that

$$[av^0] = 0 \quad [bv^0] = 0,$$

Square the residuals $v_1, v_2,$ and add, then

$$
\begin{aligned}
[vv] &= [v^0 v^0] + [\{a(1) + b(2) + \qquad \}^2] \\
&= [v^0 v^0] + [ww] \text{ suppose } .
\end{aligned}
$$

The total sum $[vv]$ may therefore be found in two parts, one from squaring the residuals of the observation equations, and the other from the corrections (1), (2), . .

32

We proceed to put $[ww]$ in a more convenient shape for computation

$$[ww] = [\{a(1) + b(2) + \quad\}^2]$$
$$= (1)\{[aa](1) + [ab](2) + \quad \}$$
$$+ (2)\{[ab](1) + [bb](2) + \quad \}$$
$$+ \quad .$$
$$= (1)\,\boxed{1} + (2)\,\boxed{2} +$$

from Eq 8, p 240

Substitute for (1), $\boxed{1}$, (2), their values from equations 8 and 10, and expand , then

$$[ww] = \{\overline{[a\text{A}]}\,I + \overline{[a\text{B}]}\,II + \quad \}I$$
$$+ \{\overline{[a\text{B}]}\,I + \overline{[b\text{B}]}\,II + \quad \}II$$
$$+$$

which may be transformed, by means of Eq 14, into the form

$$[ww] = l_0' I + l_0'' II +$$

or, as in Art 111, into the form

$$[ww] = \frac{(l_0')^2}{[a\text{A}]} + \frac{(l_0''\ 1)^2}{[b\text{B}\ 1]} + \frac{(l_0'''\ 2)^2}{[c\text{C}\ 2]} +$$

These forms may be readily computed as in Art 100

(b) Next find the weight of the given function of the adjusted values

Let the function, reduced to the linear form, be

$$dF = g_1 x + g_2 y + \tag{16}$$

where $g_1, g_2,$ are known quantities

Put for $x, y,$ their values $(x) + (1)$, $(y) + (2)$, .
and

$$dF = g_1(x) + g_2(y) + \quad + g_1(1) + g_2(2) +\quad .$$

Put for (1), (2), their values from (10), and

$$dF = g_1(x) + g_2(y) + \quad + [g\text{A}]I + [g\text{B}]II + \ldots \tag{17}$$

where l, ll, are found from the equations

$$[a\lambda]l + [ab]ll + \ldots \quad l_0' \quad 0$$
$$[ab]l + [bb]ll + \quad l_0'' \quad 0$$

Using the multipliers k_1, k_2, in order to eliminate l, ll, we have, as in Art. 111,

$$dF \quad \ldots \quad + l_0'k_1 + l_0''k_1 +$$
$$[a\lambda]k_1 \quad [ab]k_2 \quad \ldots \{l$$
$$[ab]k_1 \quad [bb]k_1 \quad \ldots \{ll \quad (18)$$

We may determine k_1, k_2, . . . so as to cause the co-efficients of l, ll, . . . to vanish, that is, so as to satisfy the equations

$$[a\lambda]k_1 + [ab]k_1 + \quad [g\lambda]$$
$$[ab]k_1 + [bb]k_1 + \ldots \quad [gb] \quad (19)$$

and then we shall have

$$dF \quad \ldots + l_0'k_1 + l_0''k_1 + \cdot$$

Substitute for l_0, l_0', from (13), and

$$dF \quad [k] + G_1(1) + G_2(v) + \ldots \quad (20)$$

where

$$G_1 \quad a k_1 \quad b k_1 \quad \ldots$$
$$G_2 \quad a k_1 \quad b k_1 \quad \ldots \quad (21)$$

We have thus expressed the function in terms of (1), (v), . . . and known quantities.

Now, since (1), (v), . . . are not independent, but are connected by the equations

$$[aa](1) + [ab](1) + \ldots \quad [al]$$
$$[ab](1) + [bb](v) + \ldots \quad [bl]$$

the problem is reduced to that already solved in Art. 101.

If, therefore, u_F is the reciprocal of the required weight,

$$u_F = [GQ] \tag{22}$$

where

$$
\begin{aligned}
.\,Q_1 &= [aa]G_1 + [a\beta]G_2 + \\
Q_2 &= [a\beta]G_1 + [\beta\beta]G_2 +
\end{aligned} \tag{23}
$$

$$. \quad . \quad . \quad . \quad .$$

the quantities $[aa]$, $[a\beta]$,   being as in the weight equations 9

Putting for G_1, G_2, ·   their values from (21) in these equations, and attending to (11), we find

$$
\begin{aligned}
Q_1 &= q_1 - \text{A}' k_1 - \text{B}' k_2 - \\
Q_2 &= q_2 - \text{A}'' k_1 - \text{B}'' k_2 -
\end{aligned} \tag{24}
$$

$$.$$

where

$$
\begin{aligned}
q_1 &= [aa]g_1 + [a\beta]g_2 + \\
q_2 &= [a\beta]g_1 + [\beta\beta]g_2 +
\end{aligned} \tag{25}
$$

Substituting in (22) for G_1, G_2, Q_1, Q_2,   their values from (21) and (24),

$$
\begin{aligned}
[GQ] = [gq] &- \; [g\text{A}]k_1 - [g\text{B}]k_2 - \\
&- \; [aq]k_1 - [bq]k_2 - \\
&+ \{[\overline{a\text{A}}]k_1 + [\overline{a\text{B}}]k_2 + \quad\; \} k_1 \\
&+ \{[\overline{a\text{B}}]k_1 + [\overline{b\text{B}}]k_2 + \quad\; \} k_2 \\
&+ . \quad\quad . \quad\quad\quad .
\end{aligned}
$$

But from (11) and (25)

$$[aq] = [g\text{A}], \; [bq] = [g\text{B}],$$

Hence, attending to (19), the above expression reduces to

$$[GQ] = [gq] - [g\text{A}]k_1 - [g\text{B}]k_2 - \;.\;.$$

or to

$$[GQ] = [gq] - \frac{[g\text{A}]^2}{[a\text{A}]} - \frac{[g\text{B } 1]^2}{[b\text{B } 1]} - \frac{[g\text{C } 2]^2}{[c\text{C } 2]} -$$

To compute $[gq]$ Multiply each of equations 25, in order, by $g_1, g_2,$, and add, and

$$[gq] = [aa]g_1g_1 + 2[a\beta]g_1g_2 + $$
$$+ [\beta\beta]g_2g_2 + $$
$$+$$

where $[aa], [a\beta],$ may be taken from the weight equations

The remaining terms of the second form of $[GQ]$ may be found from the solution of the normal equations, as shown in Art 111

Solution by Successive Approximation

115 This method of solution (due to Gauss) is of the greatest importance in adjustments involving many conditions It may be stated as follows

The condition equations may be divided into groups, and the groups solved in any order we please Each successive group will give corrections to the values furnished by the preceding groups, and the corrected values will be closer and closer approximations to the most probable values which would be found from the simultaneous solution of all the groups

For suppose we have the condition equations

$$a'v_1 + a''v_2 + \quad = l_a$$
$$b'v_1 + b''v_2 + \quad = l_b$$

$$h'v_1 + h''v_2 + \quad = l_h$$
$$k'v_1 + k''v_2 + \quad = l_k$$
$$\cdot \qquad \cdot \quad \cdot \quad \cdot$$

with

$$[pv^2] = \text{a min}$$

Let v_1', v_2',   be the values of v_1, v_2,   obtained from solving the first group alone, that is, from

$$a'v_1' + a''v_2' +   = l_a$$
$$b'v_1' + b''v_2' +   = l_b$$
.

$$[pv'^2] = \text{a min}$$

If now (v_1'), (v_o'),   are the corrections to these values resulting from the remaining equations, then since

$$[pv''] = [p \{v' + (v')\}^2]$$
$$= [pv'^2] + [p(v')^2]$$

the condition equations are reduced to

$$a'(v_1') + a''(v_2') +   = l_a'$$
$$b'(v_1') + b''(v_2') +   = l_b'$$
$$.   .   .   .   .   .$$
$$h'(v_1') + h''(v_2') +   = l_h'$$
$$k'(v_1') + k''(v_2') +   = l_k'$$
$$.   .   .   .$$

with

$$[p(v')^2] = \text{a min} ,$$

and the values of (v') found from the simultaneous solution of these equations, added to the values of v' found from the solution of the first set, would be equal to the value of v found directly

Similarly if v_1'', v_2'',   be the values of (v') obtained by solving the second set alone, and (v_1''), (v_2''),   be the corrections to these values resulting from the remaining equations, then since

$$[pv^2] = [pv'^2] + [pv''^2] + [p(v'')^2]$$

the condition equations are reduced to

$$a'(v_1'') + a''(v_2'') +   = l_a''$$
$$b'(v_1'') + b''(v_2'') +   = l_a''$$
.

$$h'(v_1'') + h''(v_2'') +   = l_h''$$
$$k'(v_1'') + k''(v_2'') +   .   = l_k''$$
.

with

$$[p(v'')^2] = a \text{ min}$$

The quantities $[pv'^2]$, $[pv''^2]$,   being positive, the minimum equation is reduced with the solution of each set, and thus we gradually approach the most probable set of values Beginning with the first set a second time, and solving through again, we should reduce the minimum equation still farther, and by continuing the process we shall finally reach the same result as that obtained from the rigorous solution   In practice the first approximation is in general close enough

It is plain that the most probable values can be found after any approximation by solving simultaneously the whole of the groups, using the values already found as approximations to these most probable values

Examples will be found in the next chapter

CHAPTER VI

116 The adjustment of the measured angles of a tri-
angulation net is a special case of the problem discussed
in the preceding chapters We assume the reader to be
acquainted with the construction and method of handling
of instruments used in measuring horizontal angles, and
shall confine ourselves to the methods of adjusting the
measured values of the angles

117 For clearness we will explain in some detail the
preliminary work necessary for the formation of the con-
dition equations In a triangulation there must be one
measured base at least, as AB Starting from this base and
measuring the angles $CAB, ABC,$ we may
compute the sides AC, BC by the ordinary
rules of trigonometry In plotting the
figure the point C can be located in but
one way, as only the measurements neces-
sary for this purpose have been made.
Similarly, by measuring the angles CBD, DCB we may
plot the position of the point D, and this can be done in but
one way If, however, the observer, while at A, had also
read the angle DAB, then the point D could have been
plotted in two ways, and we should find in almost all cases
that the lines AD, BD, CD would not intersect in the same
point In other words, in computing the length of a side
from the base we should find different values, according to
the triangles through which we passed Thus the value of
CD computed from AB would not, in general, be the same
if found from the triangles ABC, BCD, and from ABC, CAD.
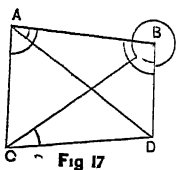
Fig 17

If the blunt angle ABD had also been measured we should have another contradiction, arising from the non-satisfaction of the relation

$$DBC + CBA + ABD = 360°$$

And not these contradictions only. For we have considered so far that in a triangle only two of the angles are measured. If in the first triangle, ABC, the third angle, BCA, were also measured, we know from spherical geometry that the three angles should satisfy the relation

$$CAB + ABC + BCA = 180° + \text{sph. excess of triangle}$$

which the measured values will not do in general. A similar discrepancy may be expected in the other triangles.

In a triangulation net, then, with a single measured base, in which the sides are to be computed from this base through the intervening triangles, we conclude that the contradictions among the measured angles may be removed and a consistent figure obtained if the angles are adjusted so as to satisfy the two classes of conditions

(1) Those arising at each station from the relations of the angles to one another at that station.

These are known as *local* conditions.

(2) Those arising from the geometrical relations necessary to form a closed figure.

(a) That the sum of the angles of each triangle in the figure should be equal to 180° increased by the spherical excess of the triangle.

(b) That the length of any side, as computed from the base, should be the same whatever route is chosen.

These are known as *general* conditions.

118. The number of conditions to be satisfied will depend on the measurements made. Each condition can be stated in the form of an equation in which the most probable values of the measured quantities are the unknowns. The number of equations being less than the number of

unknowns, an infinite number of solutions is possible   The problem before us is to select the most probable values from this infinite number of possible values

The general statement of the method of solution is this Adjust the angles so as to satisfy simultaneously the local and general conditions, that is, of all possible systems of corrections to the observed quantities which satisfy these conditions, to find that system which makes the sum of the squares of the corrections a minimum

The form of the reduction depends on the methods employed in making the observations   These methods, in general terms, are as follows. Let O in the figure be the station occupied, and A, B, C signals sighted at   The angles AOB, BOC are required   By pointing at A and then at B we find the angle AOB. Point now at B and next at C, and we have the angle BOC These two angles are *independent* of one another
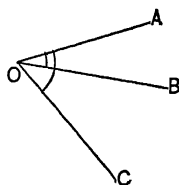
If, however, we had pointed at A, B, C in succession we should also have found the angles AOB, BOC, but they would not be independent of one another, as the reading to B enters into each

In the first method of measurement, which is known as the method of independent angles, either a repeating or a non-repeating theodolite may be used, in the second, or method of directions, a non-repeating theodolite only.

The Method of Independent Angles

119   As the case of independent angles is the simplest to reduce, we shall begin with it

A distinction must be made between angles that are independently observed and angles which are independent in the sense that no condition exists between them   Thus at the station O, above, the angles AOB, BOC, AOC might be

observed independently of one another, but we should not call them independent angles, since the condition

$$AOC = AOB + BOC.$$

must be satisfied between them. By independent angles, therefore, in the reduction, we mean those measured angles in terms of which all the measured angles can be expressed by means of the conditions connecting them. In the present case any two of the three angles AOB, BOC, AOC may be taken as independent, and the third angle would be dependent.

Angles may be measured independently either with a repeating or with a non-repeating theodolite. In primary work a non-repeating theodolite in which the graduated limb is read by microscopes furnished with micrometers is to be preferred. The method of reading an angle is as follows: The instrument, having been carefully adjusted, is directed to the left hand signal and the micrometers read. It is then directed to the other signal and the micrometers again read. The difference between these readings is called a positive single result. The whole operation is repeated in reverse order; that is, beginning with the second signal and ending with the first, giving a negative single result. The mean of these two results is called a combined result, and is free from the error arising from uniform twisting of the post or tripod on which the instrument is placed, or from "twist of station," as it is called.

The telescope is next turned 180° in azimuth and then 180° in altitude, leaving the same pivots in the same wyes, and another combined result obtained. The mean of the two combined results is free from errors of the instrument arising from imperfect adjustments for collimation, from inequality in the heights of the wyes, and from inequality of the pivots.

The distinction between these two combined results is noted in the record by "telescope direct" and "telescope reverse."

Besides those mentioned there are two kinds of system
atic error in measuring angles that deserve special atten
tion   They are the errors arising from the regular or
"periodic" errors of graduation of the horizontal limb of
the instrument, and the error from the inclination of the
limb itself to the horizon   The effects of the first may be
got rid of by the method of observation, as follows · The
reading of the limb on the first signal is changed gradually
after each pair of combined results) by some aliquot part
of the distance, or half-distance, between consecutive micro
scopes in case of two-microscope and three microscope in
struments respectively.   Thus if n is the number of pairs of

combined results desired, the changes would be $\frac{1}{n}''$ and $\frac{1}{n}''$

respectively with the instruments mentioned.   The opera

Fig 18

tion of reversal in case of a three micro
scope instrument causes each microscope
to fall at the middle of the opposite two
space, the limb remaining unchanged.
Thus if the full lines in Fig 18 represent
the positions of the microscopes with tele
scope direct, the dotted lines show their
positions with telescope reverse   In this
lies the greatest advantage of three micro
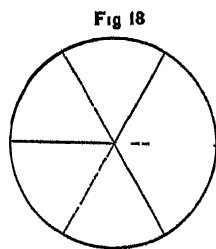
scopes over two, since with the latter, in reversing, the
microscopes simply change places with each other, without
reading on new positions of the limb.

The error arising from want of level of the horizontal
limb cannot be eliminated by the method of observation, but
with the levels which accompany a good instrument, and
with ordinary care, it will usually be less than $0'.1$.   In case,
however, of a signal having a high altitude above the hori
zon, the error from this source may be greater, and then
special care should be taken in levelling.   For an expres
sion for its influence in any case see Chauvenet's *Astronomy*,
Vol. II Art 211.

The observations should be made on at least two days

when conditions are favorable. Results obtained at different hours of a day are of more value than the same number of results obtained on different days at the same hour of the day. This is on account of variation in external conditions (direction of light, phase, distinctness, refraction, etc.)

Method for Illustration take the following example, making use of such part of it from time to time as may belong to the subject in hand, and first, after explaining the method of forming the condition equations, solve it in full.

In the triangulation of Lake Superior executed by the U. S. Engineers the following angles were measured in the quadrilateral N. Base, S. Base, Leader, Oneota:

∠N				weight	
SN					
ON					
NSO					
SN					
SO					
SON					
NO					
O					

These angles we shall denote by W_a, W_b, ... W_l respectively.

The length of the line N. Base - S. Base (Minnesota Point) is 60560" 6, and the latitude of the four stations are approximately

N. Base, 46 4'   Leader, 46 59'
S. Base, 46 44   Oneota, 46 44'

Fig 19

121. The Local Adjustment. When in a system of triangulation the horizontal angles read at a station are adjusted for all of the conditions existing among them, then these angles are said to be locally adjusted.

From the considerations set forth in Art. 117 it is readily seen that at a station only two kinds of conditions are possible—

(a) that an angle can be formed from two or more others, and

(b) that the sum of the angles round the horizon should be equal to 360°.

The second of these is included in the first, and the method of adjustment may be stated in general terms as follows

An inspection of the figure representing the angles at the station will show how all of the measured angles can be expressed in terms of a certain number of them which are independent of one another. These relations will give rise to condition equations, or *local equations*, as they are called, which may be solved as in Chapters IV. or V

Thus, if M_1, M_2, . . . M_n denote the single measured angles, and v_1, v_2, . . . v_n then most probable corrections, then if any of the angles M_h, M_k can be formed from others, we have, by equating the measured and computed values, the local condition equations,

$$M_h + v_h \quad M_1 + v_1 + M_2 + c + \ldots$$
$$M_k + v_k \quad M_1 + c_1 + M_2 + c_2 +$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

or

$$v_1 + c_2 + \quad \quad c_h \quad l_h \text{ suppose}$$
$$v_1 + c_2 + \quad \quad c_k \quad l_k \quad \text{''}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

with

$$p_1 v_1^2 + p_2 v_2^2 + \quad + p_h c_h^2 + p_k c_k^2 + \ldots + p_n c_n \quad \text{a minimum}$$

where p_1, p_2, . . . p_n denote the weights of the angles.

The solution may be in general best carried out by the method of correlates, as in Chap. V

The following special cases are of frequent occurrence:

(1) At a station O the n single angles AOB, BOC, . . . are measured, and also the sum angle AOC, to find the adjusted values of the separate angles, all of the measured values being of the same weight.

Fig 20
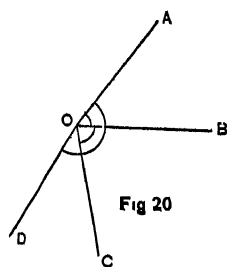
The condition equation is

$$M_1 + v_1 + M_2 + v_2 + \quad + M_{n-1} + v_{n-1} = M_n + v_n$$

or

$$v_1 + v_2 + \quad + v_{n-1} - v_n = M_n - (M_1 + M_2 + \quad + M_{n-1})$$
$$= l \text{ suppose,}$$

with

$$[v^2] = \text{a minimum}$$

The solution gives (Art 109 or 110)

$$v_1 = v_2 = \quad = -v_n = \frac{l}{n}$$

that is, *the correction to each angle is $\frac{1}{n}$ of the excess of the sum angle over the sum of the single angles, and the sign of the correction to the sum angle is opposite to that of the single angles*

(2) At a station O the n single angles AOB, BOC, LOA are measured, thus closing the horizon, to find the adjusted values of the angles

The condition equation is

$$v_1 + v_2 + \quad + v_n = 360° - (M_1 + M_2 + \quad + M_n)$$
$$= l \text{ suppose,}$$

with $\lceil pv^2 \rceil = \text{a minimum}$

The solution gives

$$v_1 = u_1 \frac{l}{[u]}$$

$$v_2 = u_2 \frac{l}{[u]}$$

where $u_1 = \frac{1}{p_1}$, $u_2 = \frac{1}{p_2}$,   , and $[u] = \left[\frac{1}{p}\right]$

If the weights are equal, then

$$v_1 = v_2 = \quad = v_n = \frac{l}{n}$$

that is, *the correction to each angle is $\frac{1}{n}$ of the excess of 360° over the sum of the measured angles*

Ex 1 The angles at station N Base close the horizon, required to adjust them

We have (Art 120)

$$M_1 + v_1 = 124' \ 00' \ 40'' \ 00 + v_1 \quad \text{weight} \cdot$$
$$M_2 + v_2 \quad 113 \quad 30' \ 05' \ 07 + v_2 \quad \text{''} \quad \text{'}$$
$$M_3 + v_3 \quad 122' \ 11' \ 15'' \ 61 + v_3 \quad \text{''} \quad 4$$

$$\text{Sum} \qquad 360' \ 00' \ 01'' \ 37 + v_1 + v_2 + v_3$$
$$\text{Theoretical sum} \quad - 360' \ 00' \ 00'' \ 00$$

Local equation is $\qquad 0 \qquad 1'' \ 37 + v_1 + v_2 + v_3$

Hence (Ex 2, Art 110)

$$v_1 = - \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} \times 1 \ 37$$
$$= - 0'' \ 64$$
$$v_2 = - 0'' \ 64$$
$$v_3 = - 0'' \ 00$$

and the adjusted angles are

$$124' \ 00' \ 40'' \ 05$$
$$113'' \ 30' \ 04'' \ 43$$
$$122'' \ 11' \ 15'' \ 52$$

Check-sum = $360'' \ 00' \ 00'' \ 00$

Ex 2 Precisely as in the preceding we may deduce at station South Hill the local equation

$$0 = 1'' \ 07 + v_4 + v_5 + v_6$$

and the adjusted angles

$$23'' \ 08' \ 05''.13$$
$$47'' \ 31' \ 19' \ 91$$
$$70° \ 39' \ 25'' \ 01$$

122 *Number of Local Equations at a Station.* If s sta-
tions are sighted at from a station that is occupied, the
number of angles necessary to be measured to determine all
of the angles that can be formed at the station occupied is
$s - 1$ If, therefore, an additional angle were measured,
its value could be determined in two ways · from the direct
measurement and from the $s - 1$ measures.   The contradic-
tion in these two values would give rise to a local (condition)
equation   If, therefore, n is the total number of angles
measured at a station, the number of local equations, as
indicated by the number of superfluous angles, is

$$n - s + 1.$$

123 **The General Adjustment.** — With a single measured base the number of conditions arising from the geometrical relations existing among the different parts of a triangulation net can be readily estimated   For if the net contains *s* stations, two are known, being the end points of the base, and *s* — 2 are to be found

Now, two angles observed at the end points of the base will determine a third point, two more observed at the end points of a line joining any two of these points will determine a fourth point, and so on   Hence to determine the *s* — 2 points, 2 (*s* — 2) angles are necessary   If, therefore, *n* is the total number of angles measured, the number of superfluous angles, that is, the number of conditions to be satisfied, is

$$n - 2(s - 2)$$

Ex In a chain of triangles, if *s* is the number of stations, show that the number of conditions to be satisfied is *s* — 2, and in a chain of quadrilaterals, with both diagonals drawn, the number of conditions is 2*s* — 4

The equations arising from these conditions are divided into two classes, angle equations and side equations

124 *The Angle Equations* —The sum of the angles of a triangle drawn on a plane surface is equal to 180°   The sum of the angles of a spherical triangle exceeds 180° by the spherical excess (ε) of the triangle, which latter is found from the relation

$$\varepsilon = \frac{\text{area of triangle}}{R^2 \sin 1''}$$

R being the radius of the sphere

From surveys carried on during the past two centuries the earth has been found to be spheroidal in form, and its dimensions have been determined within small limits   Now, a spheroidal triangle of moderate size may be computed as a spherical triangle on a tangent sphere whose radius is $\sqrt{\rho_1 \, \rho_2}$, where ρ_1, ρ_2 are the radii of curvature of the meridian and of the normal section to the meridian respectively at

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the point corresponding to the mean of the latitudes φ of
the triangle vertices

Hence we may wrap our triangulation on the spheroid
in question by conforming it to the spherical excess com
puted from the formula

$$\varepsilon \text{ (in seconds)} = \frac{ab \sin C}{2\rho_1 \rho_2 \sin 1''}$$

where a, b are two sides and C is the included angle of the
triangle

For convenience of computation we may write

$$\varepsilon = \lambda \, ab \sin C$$

when $\log \lambda$ may be tabulated for the argument φ.   The fol-
lowing table is computed with Clarke's values of the ele
ments of the terrestrial spheroid corresponding to latitudes
from $10°$ to $70°$.   The metre is the unit of length to be
used

φ	$\log \lambda$	φ	$\log \lambda$	φ	$\log \lambda$
$10°$	1 40675	$30''$	1 40547	50	1 40349
$11°$	672	31	537	51	339
$12°$	668	$32''$	528	52	329
$13°$	663	$33''$	519	53	319
$14°$	659	34	509	54	309
$15°$	1 40654	35	1 40500	55	1 40299
$16°$	649	36	491	56	289
$17°$	643	$37''$	481	57	280
$18°$	637	38	471	58	271
$19°$	631	$39''$	461	59	262
$20°$	1 40625	$40''$	1 40451	60	1 40253
$21°$	618	$41°$	441	61	244
$22°$	611	$42''$	431	62	235
$23°$	604	$43''$	420	63	226
$24°$	597	$44''$	410	64	218
$25°$	1 40589	$45''$	1 40400	65	1 40210
$26°$	581	$46°$	390	66	202
$27°$	573	$47''$	380	67	195
$28°$	564	48	369	68	188
$29°$	555	$49''$	359	69	181
$30°$	1 40547	$50°$	1 40349	70	1 40174

To find a, b, φ, a preliminary geodetic computation must
first be made of the triangulation to be adjusted, starting
from a base or from a known side   The values found from
using the unadjusted angles will be close enough for this
purpose   The latitudes need only be computed to the
nearest minute

A useful check of the excess results from the principle
that the sums of the excesses of triangles that cover the
same area should be equal   In our example the spherical
excesses of the triangles *ONS*, *LSO* will be found to be
0″ 05 and 0″ 37 respectively

In each single triangle, then, the condition required to
wrap it on the spheroid, that is, that the sum of the three
measured angles shall be equal to 180°, together with the
spherical excess, gives a condition equation [*]   This is
called an *angle equation*, or by some a *triangle equation*

Ex In the triangle N Base, S Base, Oneota, if v_3, v_4, v_8 denote the cor-
rections to the three angles, we have for the most probable values

$$ONS = 122° \ 11' \ 15'' \ 61 + v_3$$
$$NSO = 23° \ 08' \ 05'' \ 26 + v_4$$
$$SON = 34° \ 40' \ 39'' \ 66 + v_7$$

Sum $= 180° 00' 00'' 53 + v_3 + v_4 + v_7$
Theoretical sum $= 180° 00' 00'' 05 = 180° + \varepsilon$

and the angle equation is formed by equating these sums   The result is

$$v_3 + v_4 + v_7 + 0'' \ 48 = 0$$

Similarly from the triangle Lester, S Base, Oneota, the angle equation is

$$v_6 + v_7 + v_8 + v_9 + 1'' \ 10 = 0$$

125 *Number of Angle Equations in a Net* —It is to be
expected that in a triangulation net some of the lines will
be sighted over in both directions, and some in only one
direction   If these latter lines are omitted the number of
angle equations will remain unaltered   Thus in our Lake
Superior quadrilateral (Fig 19) the line *NL* has been

---

[*] We confine ourselves throughout to triangles to which Legendre's theorem is applicable
For very large triangles other formulas for spherical excess must be used   See, for example,
Helmert, *Theorieen d höheren Geodäsie*, vol 1 p 362

sighted over from N, but not from L, so that we have only two angle equations namely, those resulting from the triangles ONS, OLS, just as if the figure had been of the form of Fig 21, in which the line NL is omitted

Fig 21
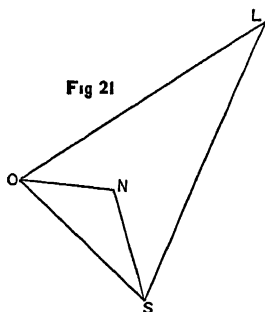
Generally, if s is the number of stations occupied, the polygon forming the outline of the net will give rise to one angle equation   Each diagonal that is drawn will form a figure, giving rise to an additional angle equation   Hence if in the net there are l_1 lines sighted over in both directions, the number of diagonals will be $l_1 - s$, and the number of angle equations

$$l_1 - s + 1$$

If l_2 of the lines are sighted over in one direction only, and l is the total number of lines in the figure, then since $l_1 = l - l_2$, the number of angle equations would be expressed by

$$l - l_2 - s + 1$$

Thus in the figure the polygon $LONS$ gives an angle equation, and the line OS gives rise to a second angle equation from either the triangle ONS or OLS We might, therefore, form the equations from either of the three sets of figures,

$$\begin{array}{ccc} LONS & LONS & ONS \\ ONS & OLS & OLS \end{array}$$

and should have respectively

$$v_1 + v_2 + v_5 + v_8 + v_9 + 3\ 06 = 0$$
$$v_3 + v_4 + v_7 + 0\ 48 = 0$$

$$v_1 + v_2 + v_5 + v_8 + v_9 + 3\ 06 = 0$$
$$v_6 + v_7 + v_8 + v_9 + 1\ 10 = 0$$

$$v_3 + v_4 + v_7 + 0\ 48 = 0$$
$$v_6 + v_7 + v_8 + v_9 + 1\ 10 = 0$$

which pairs of equations, by means of the relations already found (pp 258, 261),

$$v_1 + v_2 + v_3 + 1\ 37 = 0$$
$$v_4 + v_5 - v_6 + 1\ 07 = 0$$
$$v_3 + v_4 + v_7 + 0\ 48 = 0$$

reduce to the same two equations for each set of polygons

Ex In the quadrilateral *ABCD*, in which all of the 8 angles are measured, show that there are three independent angle equations, and that these equations may be found from the following 8 sets of figures

ABD, ABC, ACD, ABD, ABC, ABCD,
ABD, ACD, ABCD,
BDA, BCD, BCA, BCA, BCD, BCDA,
CDB, CAB, CD 1, CDB, CDA, CDBA,
DAB, DBC, DAC

Fig 22
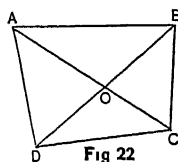
126 *The Side Equations* —In a single triangle, or in a simple chain of triangles, the length of any assigned side can be computed from a given side in but one way   When the triangles are interlaced this is not the case

Thus in Fig 21 any side can be computed from *NS* in but one way   The only condition equations apart from the local equations would be the two angle equations   But in Fig 19, in which the line *NL* is sighted over from *N*, we have the further condition that the lines *OL*, *NL*, *SL* intersect in the same point, *L*   The figure plotted from the measured values would be of the form of Fig 23

Fig 23

To express in the form of an equation the condition that the three points *L₁, L₂, L₃* must coincide, we proceed as follows
Starting from the base *NS*, we may compute *SL₁* directly
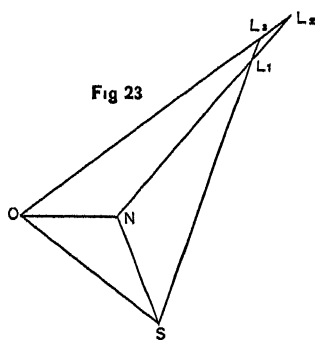
from the triangle SNL_1 and SI_1 from the triangles SON, SOL_1. This gives

$$\frac{\sin SN}{\sin SL_1} = \frac{\sin SL_1N}{\sin SNL_1}$$

$$\frac{\sin SN}{\sin SI_1} = \frac{\sin SON}{\sin SNO} \cdot \frac{\sin SI_1O}{\sin SOI}$$

But SL_1 must be equal to SI_1.

Hence the condition equation is

$$\frac{\sin SLN}{\sin SNL} = \frac{\sin SNO}{\sin SON} \cdot \frac{\sin SOL}{\sin SLO} \qquad 1$$

which is called a *side equation* or *sine equation*.

Ex. In the figure $ABCD_1D_1A$, the three angle equations

$$\begin{aligned} D_1AB + ABD_1 + BD_1A &= 1 \ \text{to} + t_1 \\ ABC + BCA + CAB &= 1 \ \text{to} + t \\ BCD_1 + CD_1B + D_1BC &= 1 \ \text{to} + t \end{aligned}$$

Fig 24
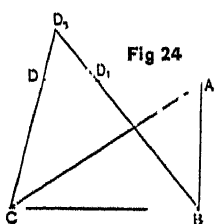
given by the triangles D_1AB, ABC, BCD_1, may be satisfied and yet the figure not be a perfect quadrilateral. Show by equating the values of BD_1 and BD, that the further condition necessary is

$$\frac{\sin DAB \sin BCA \sin CDB}{\sin BDA \sin CAB \sin BCD} \qquad 1 \qquad \blacksquare$$

The side equation

$$\frac{\sin SLN}{\sin SNL} = \frac{\sin SOL}{\sin SLO} \cdot \frac{\sin SNO}{\sin SON} \qquad 1$$

gives the identical relation

$$\frac{\sin SN}{\sin SL} = \frac{\sin SL}{\sin SO} \cdot \frac{\sin SO}{\sin SN} \qquad 1$$

Hence in forming a side equation we may proceed mechanically in this way. Write down the scheme

$$\begin{array}{ccc} SN & SL & SO \\ SL & SO & SN \end{array} \qquad 1$$

128   If the triangulation net, instead of involving quad-
rilaterals only, involves central polygons, such that, in com-
puting the lengths of the sides,
we can pass from one side to
any other through a chain of
triangles, the same process is
followed in forming the side
equations as in a quadrilateral
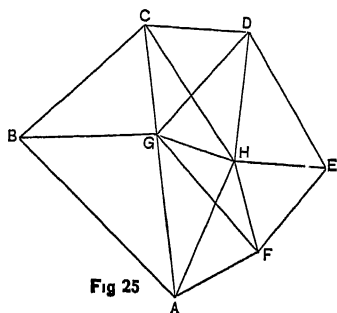
   Thus in the figure which rep-
resents part of the triangula-
tion of Lake Erie west of Buf-
falo Base there are side equa-
tions from

Fig 25

the quadrilaterals $CDHG$, $GHFA$
the pentagons $GABCH$, $HGDEF$

The scheme for the pentagonal side equation $GABCH$, for
example, would be just as in the case of a central quadri-
lateral, taking G as pole,

$$\frac{GA}{GB}\ \frac{GB}{GC}\ \frac{GC}{GH}\ \frac{GH}{GA} = 1$$

and the side equation

$$\frac{\sin GBA}{\sin GAB}\ \frac{\sin GCB}{\sin GBC}\ \frac{\sin GHC}{\sin GCH}\ \frac{\sin GAH}{\sin GHA} = 1$$

129 *Reduction to the Linear Form* —Thus far we have
considered the side equations in their rigorous form   But
in order to carry through the solution by combining them
with the other condition equations they must be reduced to
the linear form   We proceed to show how this may be
done   (See Art 7)

   Let the side equation be

$$\frac{\sin V_1}{\sin V_2}\ \frac{\sin V_3}{\sin V_4}\ = 1 \qquad\qquad (1)$$

where V_1, V_2,   denote the most probable values of the
angles   Let M_1, M_2,   denote the measured values, and

$v_1, v_2,$ the most probable corrections to these values, then the equation may be written

$$\frac{\sin (M_1 + v_1)}{\sin (M_2 + v_2)} \frac{\sin (M_3 + v_3)}{\sin (M_4 + v_4)} = 1 \qquad (2)$$

Taking the log of each side of this equation and expanding by Taylor's theorem, we have, retaining the first powers of the corrections only,

$$\log \sin M_1 + \frac{d}{dM_1}(\log \sin M_1)\, v_1$$
$$- \left\{ \log \sin M_2 + \frac{d}{dM_2}(\log \sin M_2)\, v_2 \right\} + \qquad = 0 \quad (3)$$

which may be written in two forms for computation

First, if the corrections to the angles are expressed in seconds, we may put

$$\frac{d}{dM_1}(\log \sin M_1) = \delta'$$

where δ' is the tabular difference for $1''$ for the angle M_1 in a table of log sines. Then we have

$$\delta' v_1 - \delta'' v_2 + \qquad + \log \sin M_1 - \log \sin M_2 + \qquad . = 0$$

that is,

$$[\delta v] = l \qquad (4)$$

where l is a known quantity

Secondly, we may replace

$$\frac{d}{dM_1}(\log \sin M_1) \quad \text{by} \quad \text{mod} \sin 1'' \cot M_1$$

where mod denotes the modulus of the common system of logarithms Eq 3 may then be arranged

$$\text{Cot } M_1\, v_1 - \cot M_2\, v_2 + \ldots$$
$$= \frac{1}{10^7 \,\text{mod} \sin 1''}(\log \sin M_2 - \log \sin M_1 + \quad) \quad (5)$$

if the seventh place of decimals is chosen as the unit.

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For convenience of computation there is not much to choose between the two forms   The second is perhaps, on the whole, to be preferred in ordinary triangulation work with well-shaped triangles

For the method of computing log sines and log differences for small angles or for angles near 180°, and also if a ten-place table is used, see Art 7

130 *Check Computation* —The side equation deduced from spherical triangles must also follow from the corresponding plane triangles, the angles of each spherical triangle being transformed according to Legendre's theorem, that is, for example, we should obtain the same constant term l by reducing to the linear form the equation

$$\frac{\sin SLN}{\sin SNL} \; \frac{\sin SOL}{\sin SLO} \; \frac{\sin SNO}{\sin SON} = 1$$

or the equation

$$\frac{\sin (SLN - \tfrac{\varepsilon_1}{3})}{\sin (SNL - \tfrac{\varepsilon_1}{3})} \; \frac{\sin (SOL - \tfrac{\varepsilon_2}{3})}{\sin (SLO - \tfrac{\varepsilon_2}{3})} \; \frac{\sin (SNO - \tfrac{\varepsilon_3}{3})}{\sin (SON - \tfrac{\varepsilon_3}{3})} = 1$$

where ε_1, ε_2, ε_3 denote the spherical excesses of the triangles *SNL, SOL,* and *SON* respectively

It affords a check of the accuracy of the numerical work to compute the side equation with both the spherical angles and the plane angles   It is evidently simpler to use the spherical angles, so that if but a single computation is to be made they should be chosen

For a check of the coefficients of the corrections we have, by expanding the second equation by Taylor's theorem, the relations

$$\varepsilon_1(\delta' - \delta'') + \varepsilon_2(\delta''' - \delta'''') + \varepsilon_3(\delta''''' - \delta'''''') = 0$$

or

$$\varepsilon_1(\cot SLN - \cot SNL) + \varepsilon_2(\cot SOL - \cot SLO) + \varepsilon_3(\cot SNO - \cot SON) = 0$$

for the first and second forms of reduction respectively

This useful check is given by Andrae

7. The quadrilateral N Base, S Base, Oneota, Lester (Fig 19)
Take the pole at Lester
We have the scheme

$$\begin{array}{c} LS\ LN\ LO \\ LN\ LO\ LS \end{array} \quad {}^{1}$$

from which we write down the side equation

$$\begin{array}{c} \sin LNS\ \sin LON\ \sin LSO \\ \sin LSN\ \sin LNO\ \sin LOS \end{array} \quad {}^{1}$$

that is,

$$\begin{array}{c} \sin(M + \ ) \sin(M_i + \ ) \qquad \sin(M_i + \omega) \\ \sin(M + \ ) \sin(M_i + \ ) \sin(M_i + M_i + \ + \omega) \end{array} \quad {}^{1}$$

First Form of Reduction

log sin (   )   9 9615969,7   9,22
log sin (   )   9 8390903,4 + 21,05
log sin (,0   )   9 9747056,9 + 7,39

                                        530,0

log sin (   )   9 8677850,5 + 19,25
log sin (   )   9 9177170,7   14,90
log sin (   )   9 9911159,4 + 4,30(   )

                                         10,0

Hence the side equation in the linear form is

14 90   9 ,   19 25   + 7 39   + 30 , + 17/05   + 20 0   0

the unit being the seventh place of decimals

Check of the constant term by computing the log sines after deducting
from each angle ½ of the spherical excess of the triangle to which it belongs

Angle	Log Sin	Angle	Log Sin
	9 9615969,4	47 31' 20" 11	9.8677858,0
	9 8390905,5	121 00' 10" 06	9 9177170,7
	9 9747056,60	75 07' 05" 93	9 9911179,8
	,		505,7
	+ 30,1		

agreeing closely with the value found from the spherical angles.

Check of the coefficients

0 30(9 25   19 90) + 0 13(21 04 + 14 90) + 0 3/(7.39   + 30)   + 0 08

This check would have been closer had the spherical excesses been carried
out to three places of decimals   We have taken 0.19, 0 05, and 0".12 for the

excesses of the single triangles LAS, OAS, LSO, and ... for the ...
triangle LSO The first two are more nearly ... to ...

<center>Second Form of Remark</center>

Log Sin		Log Sin	
9 9618969,7	0 4380 ...	9 6... 9	...
9 8399903,4	+ 1 0147.5	9 917.4,0 ...	...
9 9747056,9	+ 0 3510...	9 9911 ...	...
	530,0		
	510,0		
	20,0	log 1 30103	
$\frac{1}{10^7 \bmod \sin 1''}$		log 8 67661	
	9 97767	...	

and the side equation is

0 6786 c_1 0 4380 c_2 0 0156 c_3 + 0 3510 ...   ...

This result may be checked in the same way as in the first form

In reducing a side equation to the linear form the tenth ... of the ... rections should be carried out to one place of decimal   In the ... lute term   This for a short computation would be unnecessary ... reduction of an extensive triangulation net it is rendered ... the accumulation of errors from the dropping of the last figure ... quotients

It will be noticed that in the preceding example we have carried out the log. sines to 8 places of decimal, the seventh place being the unit   This is amply sufficient, in primary work, for our present methods of measurement, as already indicated in Art. 34.   Indeed, it is in general sufficient to carry them to 7 places of decimals only, an ordinary 7 place table being used   As there is no 8 place table published, the labor of forming the log. sines with a 10 place table, and then cutting down the results to 8 places, is, in very many cases, hardly justified by the extra precision attained

For a secondary triangulation 6 places of decimal, and for a tertiary triangulation 5 places, may be used.

* On the Coast Survey and Lake Survey the practice in primary triangulation is to carry out the log sines to 10 places of decimals. On the English ... carried out to 10 places, but on the more modern Great Trigonometrical Survey ... places only   In the triangulation of Denmark, Andræ used 7 places, as is the ... lowed by the Prussian, Italian, and other modern European surveys.

131 We have seen that the coefficients of the corrections in a side equation are given by the differences for 1 of the log sines of the angles, or by the cotangents of the angles that enter. Now, since an equable distribution of errors arising from the approximate computation is best attained by securing the greatest possible equality of coefficients throughout the condition equations, and since the coefficients of the corrections in the angle equations are ± 1 or -1, it follows that it would be most convenient to put the side equations on the same footing as the angle equations. To do this we may divide the side equation by such a number as will make the average value of the coefficients equal to unity. This, for angles ordinarily met with in triangulation, would be effected by taking the sixth place of decimals as the unit in the side equation. Thus in our example, dividing by 10, which is approximately the mean of the coefficients, and which amounts to the same thing as expressing the log differences in units of the sixth place of decimals, the equation may be written

1.484   0.905   1.036 + 0.740   0.436 + 1.770 + 2.00   0

It would have been equally correct to have multiplied each of the angle equations by 10, and so have put them on the same footing as the side equations. Dividing the side equations is, however, to be preferred, as the coefficients are made smaller throughout, and the formation and solution of the normal equations is consequently easier.

A striking difference between condition equations and observation equations is here brought out. As a condition equation expresses a rigorous relation among the observed quantities altogether independent of observation, it may be multiplied or divided by any number without affecting that relation; with an observation equation, on the other hand, the effect would be to increase or diminish its weight. (Compare Art. 58.)

132. *Position of Pole.* In a quadrilateral, taking any of the vertices as pole, the conclusion was reached in Art. 127 that

any one of the resulting forms of side equation was as
good as any other in satisfying the conditions imposed
But when a side equation is reduced to the linear form
and is no longer rigorous the question deserves further
notice

Two points are to be considered precision of results and
ease of computation As regards the first, since the differ-
ences in a table of log. sines are more sharply defined for
small angles, and these differences are the coefficients of
the unknowns in the side equation, it follows that in practice
that vertex should be chosen which allows the introduction
of the acutest angles into the side equation

Labor of computation will be saved by choosing the pole
so that as few sine terms as possible enter Thus by choos-
ing the pole at O, the intersection of the diagonals (Fig. 2),
the side equation would contain 8 terms, whereas if taken
at any of the vertices only 6 terms would enter. Also,
other things being equal, we should choose that pole which
introduces the smallest number of unknowns into the equa-
tion, for then the normal equations would be more easily
formed

If the approximate form of solution in Art. 115 is em-
ployed it is advantageous to choose the pole at the inter-
section, O, of the diagonals, as will be seen in the sequel

133 *Number of Side Equations in a Net.* A line being
taken as a base, its extremities are known To fix a third
point we must know the other two sides of the triangle of
which this point is to be the vertex Hence if we have a
net of triangles connecting s stations, two of the stations
being the ends of the base, we must have, in order to fix
the figure, $2(s - 2)$ lines besides the base, that is, $2s - 3$
lines in all

Starting from the base, each line in the figure can be
computed in but one way, but any additional line, whether
observed over in one or both directions, can be computed
in two ways, and therefore gives rise to a side equation
If, then, the total number of lines in the figure is l, the num-

ber of side equations, as indicated by the number of super-
fluous lines, is

$$l - 2s + 3$$

134 *Check of the Total Number of Conditions* —Leaving
local equations out of account, if l is the number of lines in
a figure sighted over in both directions, and s the number
of stations, the total number of angles in the figure is $2l - s$
If l_1 of these lines have been sighted over in one direction
only, the number of angles is reduced to $2l - l_1 - s$

Now, the number of angle equations in the figure is

$$l - l_1 - s + 1$$

and the number of side equations is

$$l - 2s + 3$$

the total number of condition equations is

$$2l - l_1 - 3s + 4, \quad \text{that is, } n - 2s + 4$$

where n is the number of angles in the figure

But we have seen in Art 123 that the total number of
conditions to be satisfied among the same n angles is

$$n - 2s + 4$$

We conclude, therefore, that the conditions are completely
covered by the angle and side equations

135 **Manner of Selecting the Angle and Side
Equations.**—In the selection of the angle and side equa-
tions in a triangulation net we have two dangers to guard
against first, that we omit no necessary conditions, and,
second, that we admit no unnecessary ones   The rule
usually followed is to start from some line as base, and plot,
the figure proceeding from station to station,
writing down the conditions that express
the connections of each station to the net as
the net grows
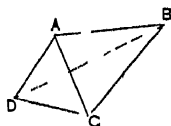
For example, let Fig 27 represent a tri-
angulation net   Taking AB as base (Fig 26), a third station,
C, gives an angle equation from the triangle ABC A

Fig 26

fourth station, D, gives in addition angle equations from ABD, ACD, side equation from BCD. A fifth point, E, gives in addition (Fig. 27) angle equations from ABE, ADE, side equation from $ABDE$.
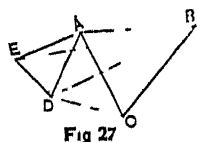
We have thus in the complete figure (Fig. 27) 5 angle equations, and 4 side equations.

Fig 27

Check Number of stations   5
         Number of lines   9
         Number of angle equations   9 − 5 + 1   5
     and Number of side equations   9 − 5 + 1   4

As an illustration of the difficulties which may arise in selecting the angle and side equations, let us take the triangulation around the Chicago Base (1873). It is represented in the figure.

From the rules laid down in Arts 125, 133 it follows that there are in the adjustment 10 angle equations and 8 side equations.

The peculiarity of the system is that the station F is very close to the base line DE. Thus the angle equation from the triangle DEF

Fig. 28

$$EDF \quad (o)'' (o)' (o)''.815 + c_1$$
$$FED \quad (o)'' (o)' \quad 1''.185 + c_2$$
$$DFE \quad 179'' 59' 56''.733 + c_3 + c_4$$
$$\quad 179'' 59' 58''.733 + c_1 + c_2 + c_3 + c_4$$
$$\quad 180'' (o)' (o)''.(oo)$$

$$\therefore \quad 0 \quad 1''.267 + c_1 + c_2 + c_3 + c_4$$

In the selection of the side equations it is advisable to avoid those quadrilaterals in the figure which are entangled
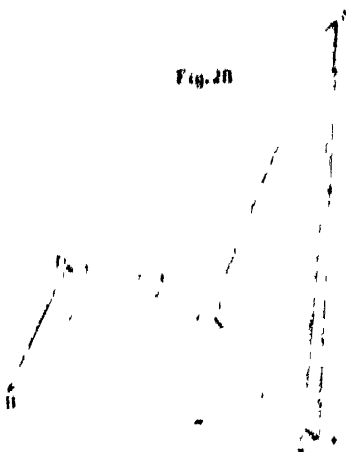
with the above triangle, that is, the quadrilaterals $AFDE$, $EDFI$, $GDFI$. For example, if we take the quadrilateral $GDFF$ we have, in units of the seventh place of decimals, pole at G, the side equation

$$s1_{zz}{}^{\prime}b \quad \cdots \quad \cdots \cdots 35, \quad +0{\cdot}0000c + z_i)$$
$$+ {}^{90}480{\cdot}3c, \quad \cdots 0003(c + c_i) \quad +0{\cdot}13$$

Now, since the coefficients of c, c_i, c, z_i, are each less than 5 in the third place, the equation is nearly the same as

$$s1_{zz}{}^{\prime}b \quad +0{\cdot}0740c + c_i) + {}^{90}480z_i \quad +0{\cdot}13$$

or dividing by 3 and replacing $c + c_i$ by $c_i + c_0$

$$1{\cdot}7\cdots + 1{\cdot}083c + c_i) + 0{\cdot}65\,17, \quad 1$$

which is nearly the same as the angle equation from the triangle DFF.

Similarly the quadrilaterals $IFDE$, $BDFF$ give respectively, neglecting coefficients less than 5 in the fifth place,

$$1{\cdot}70c, +0{\cdot}00c + 0{\cdot}77c + c_i) \quad 0{\cdot}700$$
$$c \cdot 00c_i + {}^{9}p0c + 1{\cdot}333(c + c_i) \quad 0{\cdot}830$$

both of which express approximately the same relations among the angles of the small triangle DFF as the angle equation formed from this triangle. We therefore conclude that in the formation of the condition equations other quadrilaterals than these should be chosen.

136. Again, in selecting the side equations in a net care must be taken that only independent conditions are chosen. Thus, in Fig. 43 we might have chosen the following eight quadrilaterals from which to form the side equations

$$AGBD, \quad AGDF, \quad AGBF, \quad BDFG,$$
$$BIFG, \quad IGDF, \quad AGBF, \quad BDFG$$

A careful examination will show that these figures are not independent. For, taking the four, $AGBD$, $AGDF$, $AGBF$,

BDEG, and choosing *G* as pole, which is common to all, we have the side equations

$$\frac{\sin ADG}{\sin DAG} \frac{\sin DBG}{\sin BDG} \frac{\sin BAG}{\sin ABG} = 1$$

$$\frac{\sin DAG}{\sin ADG} \frac{\sin EDG}{\sin DEG} \frac{\sin AEG}{\sin EAG} = 1$$

$$\frac{\sin ABG}{\sin BAG} \frac{\sin BEG}{\sin EBG} \frac{\sin EAG}{\sin AEG} = 1$$

$$\frac{\sin BDG}{\sin DBG} \frac{\sin DEG}{\sin EDG} \frac{\sin EBG}{\sin BEG} = 1$$

which equations multiplied together reduce to the identical form

$$1 = 1$$

showing that from any three the fourth may be found

The entrance of mutually dependent conditions would, however, be detected in the course of the solution of the normal equations, as we should arrive at two identical equations, or, in other words, one of the correlates would become indeterminate

If the rule given on p 273 is followed closely this repetition of conditions will hardly occur

Fig 29   Fig 30   Fig 31

Fig 32   Fig 33   Fig 34
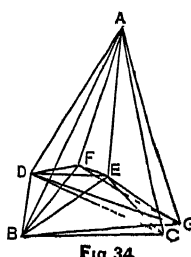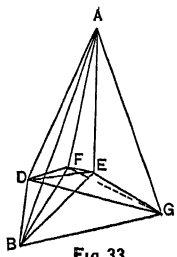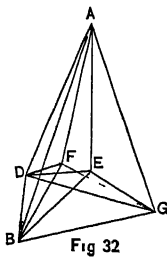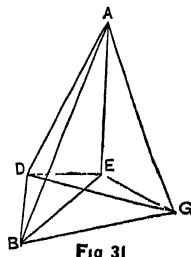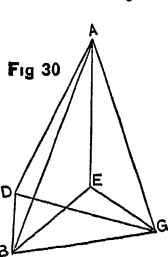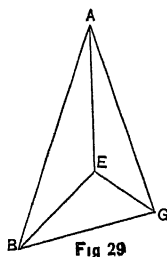
137. In the example chosen (Fig. 28) the selection of the angle and side equations may be made in the following order, proceeding from the line *IG*, and adding station to station and line to line till the complete figure is reached

		Angle equations	Side equations
From Fig. 29,	Triang.	*IGB*	
	"	*AFB*	
	"	*BIG*	
			Quad *AGBE*
From Fig. 30 in addition,	"	*DAG*	
	"	*BDG*	
			" *AGBD*
From Fig. 31 in addition,	"	*DAF*	
			" *AGDE*
From Fig. 32 in addition,	"	*DAF*	
			" *AGDF*
	"	*BDF*	
			" *BDFG*
From Fig. 33 in addition,	"	*BEF*	
			" *BEFG*
From Fig. 34 in addition,	"	*BAE*	
			" *ACBE*
			" *CBDE*

In all 10 angle equations and 8 side equations, as should be (p. 274).

138. We finally notice the *arrangement* for solution of the condition equations in the net adjustment. On first thoughts it might seem that it would be well to arrange the angle equations and side equations in two separate sets, and so carry them forward for solution. This was done in some of the older work.* The only objection to doing it is that the process of finding the corrections is more troublesome. Ex

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* See for an example *Lesson in the art of Least Squares*, &c. of *Meridian*, by Sir Thomas Maclear, Cape of ...

perience shows that the solution of a set of normal equa-
tions is much facilitated if the coefficients are arranged in
the steps of a stair rather than irregularly. Thus

$$[aa|1 \quad [ab|1$$
$$[ab|1 \quad [bb|r \quad ...]$$
$$[...|r \quad ...]$$

is a more convenient form for solution than

$$[aa|1 \qquad [ab|1 \quad ...]$$
$$[...| \quad [b...| \quad ...]$$
$$[ab|1 \quad [...|. \quad [bb|1 \quad ...]$$

The condition equations should therefore be so arranged
that, as far as possible (and it cannot always be done at the
first trial), the normal equations will fall in the first of the
above forms rather than the second. A good rule is to
begin with an angle equation, proceeding from triangle to
triangle until the points gone over are covered by a side
equation, and then introduce it. Continue the process with
the remaining sets of triangles.

As to the angle equations themselves, it matters but
little which triangles are taken to form them. It is best,

Fig 35

however, to avoid forming a
very small angle, such as DHJ
in Fig. 35, and, where such
very small occur, to avoid
angles involving an intermediate
diately contiguous to these small
angles.

130. In the explanation of the
formation of the side equations
we have assumed that, in the com-
putation of one side from another,
assumed as base, by different

routes we proceed through chains of triangles. The
omission of certain lines in the measurement may make it
necessary to proceed through polygons and then the for-
mation of the side equations becomes more complicated.
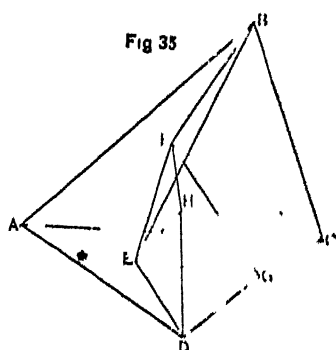
A good illustration occurs in the triangulation of Lake Superior near Keweenaw Base, as shown in Fig 35

Here there are 18 lines and 8 stations, requiring 5 side equations   Proceeding from the line DG and adding point to point, we form the first four side equations from the quadrilaterals

$$HGDI, HGFE, HEFC, CFFB$$

With these there is no difficulty

The fifth side equation is furnished by the pentagon $HCHD$ We cannot compute directly any specified side of this pentagon from another side through triangles only   A little artifice, however, will enable us to do this   Suppose the line CD drawn   Call the angle GDC i and FCD r This new line CD gives an additional side equation.   We take the two pentagons $HCFD, DFGHC$.

From the first pentagon, $HCFD$,

$$\frac{CF}{CD} \cdot \frac{CD}{CI} \cdot \frac{CI}{CB} \cdot \frac{CB}{CF} \quad _{1}$$

or

$$\frac{\sin(FDG + i)}{\sin CFD} \cdot \frac{\sin CID}{\sin(IDG + i)} \cdot \frac{\sin CBI}{\sin CIB} \cdot \frac{\sin CFB}{\sin CBF} \quad _{1} \quad (1)$$

and from the second pentagon, $DFGFC$, pole at F,

$$\frac{FD}{FG} \cdot \frac{FG}{FF} \cdot \frac{FF}{FG} \cdot \frac{FC}{FD} \quad _{1}$$

or

$$\frac{\sin FGH}{\sin FDG} \cdot \frac{\sin FHG}{\sin FGF} \cdot \frac{\sin FCF}{\sin FFC} \cdot \frac{\sin(FDG + i)}{\sin r} \quad _{1} \quad (2)$$

Now, r can be expressed in terms of i from the triangle CDF, thus

$$r + CFD + FDG + i = 180 + e \quad (3)$$

where e is the spherical excess of the triangle CDF

Eliminating x from equations 1, 2, 3, and the required side equation results

To find it write (1) in the form

$$\cot (EDG + x)$$
$$= \frac{1}{\sin ADE} \; \frac{\sin CAD}{\sin CED} \; \frac{\sin ABC}{\sin BAC} \; \frac{\sin BEC}{\sin CBE} - \cot ADE$$

from which the value of x follows at once

Hence since y is known from (3), all the angles in (2) are known, and the solution is finished in the usual way

If chains of triangles intersect so as to form a closed polygon, this method may be employed in simple cases Such forms are, however, in general better treated by other methods, which will be found in works on the higher geodesy

140 Ex —Adjustment of the Quadrilateral NSOL (Fig 19)

The method of forming the condition equations having now been explained, we are ready to adjust the quadrilateral $NSOL$, as promised in Art 120

The condition equations have all been formed in the preceding sections Collecting them, we have

Local equations (Ex 1, 2, Art 121)

$$v_1 + v_2 + v_3 = - 1\ 37$$
$$v_4 + v_5 - v_6 = - 1\ 07$$

Angle equations (Ex Art 124)

$$v_3 + v_4 + v_7 \quad = - 0\ 48$$
$$v_6 + v_7 + v_8 + v_9 = - 1\ 10$$

Side equation, the unit being the sixth place of decimals (Ex Art 130),

$$1\ 43v_1 - 0\ 92v_2 - 1\ 93v_5 + 0\ 74v_6 - 0\ 43v_7 + 1\ 77v_8 = - 2\ 00$$

The methods of solution have been explained in Chap V , and we shall proceed in the order there given for the four forms

FIRST SOLUTION—METHOD OF INDEPENDENT UNKNOWNS

There being 9 unknowns and 5 condition equations connecting them, there must be 4 independent unknowns. We shall choose v_1, v_2, v_4, v_6. Expressing all of the unknowns in terms of these four, we write the equations in the form of observation equations, as follows (see Art. 109)

						weight	2
v_1						"	2
v_2						"	2
	v_1	v_2			-1.37	"	$1\frac{1}{4}$
			v_4			"	23
v_6				v_6		"	6
			$v_4 +$	$v_6 + 1.07$		"	7
	$v_2 +$	v_4	v_4	$+ 0.89$		"	31
	$0.505 v_1 + 0.703 v_2$	$0.661 v_4 + 0.672 v_6$	$+ 3.61$			"	1
	$0.435 v_1 + 0.703 v_2 + 0.661 v_4 + 0.672 v_6$	$+ 6.99$				"	8

(1)

Hence the normal equations

v_1	v_2	v_4	v_6	Const	
$+ 18.84$	$+ 50.70$	3.93	$+ 5.41$	53.45	
	$+ 7?.15$	10.83	$+ 21.00$	60.70	
		$+ 64.93$	2.29	$+ 25.18$	
			$+ 35.82$	29.30	
				$+ 83.70$	$[pll]$

Solving these equations (page 284), we have the values of the corrections

$$v_1 \quad 0''.82 \qquad v_4 \quad -0''.29$$
$$v_2 \quad 0''.36 \qquad v_6 \quad 0''.47$$

and thence from the condition equations

$$v_3 \quad 0''.19 \qquad v_8 \quad 1''.33$$
$$v_5 \quad +0''.38 \qquad v_9 \quad -0''.08$$
$$v_7 \quad 0''.07$$

These corrections applied to the measured values of the angles give the most probable values as follows

M_1	17°	00'	30".87		M_6	76°	39'	21".08
M_2	11°	39'	04".71		M_7	34°	40'	30".50
M_3	122°	11'	18".42		M_8	13°	16'	05".07
M_4	23°	08'	05".04		M_9	30°	53'	30".73
M_5	47°	31'	19".94					

The Precision of the Adjusted Values —(a) To find the m s e of an obser-vation of the unit of weight (Arts 99, 101)

From the above values of the residuals v

$$[pvv] = 7\ 53$$

Check of $[p\text{-}v]$ Carrying through the solution of the normal equations the extra column required by the sum $[pll]$, we find (page 284)

$$[pvv] = 7\ 54$$

Hence

$$\mu = \sqrt{\frac{7\ 54}{9 - 4}}$$

$$= \pm\ 1''\ 23$$

(b) To find the weight and m s e of the adjusted value of an angle

Take the angle NLS Proceeding as in Art 101, we have

$$F = NLS$$
$$= 180 + \varepsilon - (M_2 + v_2 + M_5 + v_5)$$
$$dF = - v_2 - v_5$$

Hence from the extra column, the sixth, carried through the solution of the normal equations (page 284),

$$u_F = 0\ 053$$

and therefore

$$\mu_F = 1\ 23\ \sqrt{0\ 053}$$
$$= 0''\ 28$$

(c) To find the weight and m s e of the adjusted value of a side, the base, NS, being supposed to be free from error

Let us take the side OL We have

$$F = OL$$

$$= NS\frac{\sin ONS}{\sin SON}\frac{\sin LSO}{\sin OLS}$$

$$= NS\frac{\sin (M_3 + v_3)}{\sin (M_7 + v_7)}\frac{\sin (M_6 + v_6)}{\sin (M_9 + v_9)}$$

For check we shall proceed in two ways

(1) Expand Z directly, then

$$Z = \left(\frac{\partial l}{\partial M_1} + \frac{\partial l}{\partial M_2} + \cdots + \frac{\partial h}{\partial M} + \cdots + \frac{\partial h}{\partial M_n} \right) \sin 1$$

by substituting for $\ldots$, their values from equations 1.

Carry through the solution of the normal equations the extra column required by the z coefficients (and see page 41)

Hence

$$u_1 = \ldots$$

$$u_2 = \ldots$$

(2) Take logs of both members of the equation, then

$$\log l = \log N's + \log \sin l M \quad + \log \sin (M_a + \ldots)$$
$$\log \sin (M \quad + \ldots) \quad \log \sin (M_b + \ldots)$$

But since $N's$ is constant, we have, in units of the sixth place of decimals,

$$d \log l = \ldots$$
$$\qquad \text{from equations 1}$$

Hence from the last column added to the solution of the normal equations,

$$u_{1\ldots} = \ldots \text{ in unit of the sixth place of decimals}$$

Also,

$$u_{2\ldots} = \ldots$$
$$\qquad \text{in unit of the sixth place of decimals}$$

Now, since

$$d \log l = \ldots \text{ unit}$$

and

$$l = \ldots$$
$$u_l = \ldots$$

The solution of the normal equations, with the extra columns required by the weight determinations, is as follows

v_1	v_2	v_4	v_5	l	f(angle)	f(side)	f(side)
+ 48 83	+ 50 70	− 32 93	+ 5 44	− 53 45	− 1	− 0 007	− 0 18
	+ 72 45	− 40 83	+ 21 09	− 69 70		+ 0 171	+ 4 50
		+ 64 93	− 2 29	+ 28 18		+ 0 056	+ 1 15
			+ 35 82	− 29 30	− 1	+ 0 253	+ 6 63
			[pll] =	+ 83 79			
+ 1	+ 1 0383	− 0 6743	+ 0 1114	− 1 0916		− 0 0001	− 0 0037
	+ 19 8082	− 6 6430	+ 18 4420	+ 14 2098	− 1	+ 0 1761	+ 4 6871
		+ 42 7253	+ 1 3784	+ 7 8652		+ 0 0517	+ 1 4285
			+ 35 2140	+ −3 3454	− 1	+ 0 2535	+ 6 6501
				+ 25 2836	0	0	+ 0 0017
	+ 1	− 0 3354	+ 0 9310	+ 0 7176	+ 0 0505	+ 0 0089	+ 0 2466
		+ 40 4972	+ 7 5630	+ 12 6322	− 0 3455	+ 0 1118	+ 2 9002
			+ 18 0445	+ 10 1114	− 0 0687	+ 0 0894	+ 2 2867
				+ 15 0910	+ 0 0505	+ 0 0016	+ 1 1089
		+ 1	+ 0 1868	+ 0 3119	− 0 0083	+ 0 0027	+ 0 0710
			+ 16 6317	+ 7 7525	− 0 0057	+ 0 0690	+ 1 7152
				+ 11 1510	+ 0 0028	+ 0 0003	+ 0 2076
			+ 1	+ 0 4661	− 0 0001	+ 0 0003	+ 0 1019
			[pvv] =	+ 7 5376	0	0	+ 0 1832
					0 0000	0 0000	0 0007
					0 0505	0 0016	1 1089
					0 0028	0 0003	0 2076
					0 0000	0 0000	0 1832
					0 0523	0 0019	1 5004
					= u_1	= u_k	= u_p

The solution has been carried through to four places of decimals, on account of loss of accuracy arising from dropping figures in multiplications   The resulting values of the corrections have been cut down to two places of decimals   The work was done with a machine, as explained on p 161, the reciprocals of the diagonal terms being used so as to avoid divisions   Thus the first reciprocal is 0 02048

SECOND SOLUTION—METHOD OF CORRELATES

Arranging the condition equations in tabular form, we have

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	
weights 2	2	14	23	6	7	31	1	8	
+ 1 43	− 0 92			− 1 93	+ 0 71	− 0 43	+ 1 77		− 2 00
+ 1	+ 1	+ 1				+ 1			− 1 27
		+ 1	+ 1		− 1				− 0 48
			+ 1	+ 1	+ 1	+ 1			− 1 07
				− 1		+ 1	+ 1	+ 1	− 1 10

The Correlate Equations

	I	II	III	IV	V
$2v_1 =$	$+1\ 43$	$+1$			
$2v_2 =$	$-0\ 92$	$+1$			
$14v_3 =$		$+1$	$+1$		
$23v_4 =$			$+1$	$+1$	
$67v_5 =$	$-1\ 93$			$+1$	
$7v_6 =$	$+0\ 74$			-1	$+1$
$31v_7 =$	$-0\ 43$		$+1$		$+1$
$7v_8 =$	$+1\ 77$				$+1$
$8v_9 =$					$+1$

The Normal Equations

I	II	III	IV	V	l
$+5\ 281$	$+0\ 255$	$-0\ 014$	$-0\ 427$	$+1\ 862$	$-2\ 00$
	$+1\ 071$	$+0\ 071$			$-1\ 37$
		$+0\ 147$	$+0\ 043$	$+0\ 032$	$-0\ 48$
			$+0\ 353$	$-0\ 143$	$-1\ 07$
				$+1\ 300$	$-1\ 10$

The solution of these equations gives (see page 287)

$$I = -0\ 3973$$
$$II = -1\ 0749$$
$$III = -1\ 6006$$
$$IV = -3\ 5721$$
$$V = -0\ 6301$$

Substituting these values in the correlate equations, the same values of the corrections result as before. Also,

$$[pvv] = 7\ 53$$

The Precision —(a) To find the m s e μ of an observation of weight unity
From the values of v we find directly

$$[pvv] = 7\ 53$$

Checks of $[pvv]$ These are worked out in the solution of the normal equations on page 287, according to the formulas of Art 111 and give 7 54 and 7 55 respectively

Hence taking the mean, $[pvv] = 7\ 54$, and the number of conditions being 5,

$$\mu = \sqrt{\frac{7\ 54}{5}}$$
$$= 1''\ 23 \text{ as before}$$

Compare Ex 2, Art 111
(b) To find the weight and m s e of the adjusted value of an angle,
Take the angle NLS

$$dF = -v_2 - v_5$$

From the values of u, a, b,   in the condition equations in connection with the values of f given by this function, we have

$$[uaf] = +0\ 782 \qquad [udf] = -0\ 167$$
$$[ubf] = -0\ 500 \qquad [uef] =\quad 0$$
$$[ucf] =\quad 0 \qquad [uff] = +0\ 667$$

Hence from the seventh column in the solution of the normal equations (page 287),

$$u_F = 0\ 053$$

and

$$\mu_F = 1\ 23\ \sqrt{0\ 053}$$
$$= 0''\ 28$$

Compare Ex 4, Art 111
(c) To find the weight and mean-square error of the adjusted value of a side, the base being free from error
Take the side Oneota–Lester
As in (c), page 282, we have

$$dF = -0\ 0505v_3 + 0\ 0282v_6 - 0\ 1160v_7 - 0\ 1342v_9$$

Also from the condition equations

$$[uaf] = +0\ 0046 \qquad [udf] = -0\ 0040$$
$$[ubf] = -0\ 0036 \qquad [uef] = -0\ 0165$$
$$[ucf] = -0\ 0073 \qquad [uff] = +0\ 0030$$

Hence from the eighth column in the solution of the normal equations,

$$u_F = 0\ 0023$$

and finally,

$$\mu_F = 1\ 23\ \sqrt{0023}$$

$$= 0^m\ 06$$

Solution of the Normal Equations

I	II	III	IV	V	l	f (angle)	f (side)
+ 5 284	1 0 255 + 1 071	− 0 014 + 0 071 + 0 147	− 0 427 + 0 043 + 0 353	+ 1 862 + 0 032 − 0 143 + 1 300	− 2 00 − 1 37 − 0 48 − 1 07 − 1 10	+ 0 782 − 0 500 − 0 167 + 0 667	+ 0 0046 − 0 0036 − 0 0073 − 0 0040 − 0 0165 + 0 0030
+ 1	+ 0 0483	− 0 0026	− 0 0808	+ 0 3524	− 0 3765	+ 0 1480	+ 0 0009
	1 1 0587	1 0 0717 + 0 1470	+ 0 0206 1 0 0419 + 0 3185	− 0 0902 1 0 0369 + 0 0075 1 0 6438	− 1 2731 − 0 4853 − 1 2316 − 0 3952 + 0 7570	− 0 5377 + 0 0021 − 0 1045 − 0 2756 + 0 5510	− 0 0038 − 0 0073 − 0 0036 − 0 0182
		+ 1	+ 0 0677	1 0 0201	− 0 0851	− 1 2025	− 0 5078	− 0 0036
			1 0 1121	+ 0 0401 1 0 3181	1 0 0434 + 0 0093 1 0 6361	− 0 3991 − 1 2068 − 0 5037 + 1 5409	1 0 0285 − 0 0940 − 0 4214 + 0 2780	− 0 0070 − 0 0035 − 0 0185 + 0 0030
			+ 1	+ 0 2843	1 0 3054	− 2 8086	+ 0 2709	− 0 0493
				1 0 3066	0 0040 1 0 6228	− 1 0433 − 0 3818 + 1 1209	− 0 1039 − 0 3332 + 0 4676	− 0 0015 − 0 0164 + 0 0027
Values of the Unknowns			+ 1	− 0 0098	− 3 5659	− 0 3389	− 0 0049
	I = 0 3973 II = − 1 0749 III = 1 6006 IV = − 3 5721 V = − 0 6301			1 0 6228	− 0 3924 + 3 8986	− 0 4342 1 0 2324	− 0 0164 + 0 0027
				1 1	0 6301 = V 1 0 4472	− 0 5366 + 0 0531	− 0 0264 + 0 0023
						$= u_F$	$= u_F$

I   × l' = − 0 3973 × − 2 00 = 0 79   0 7570
II × l'' = − 1 0749 × − 1 37 = 1 47   1 5309
III × l''' = − 1 6006 × − 0 48 = 0 77   1 1209
IV × l'''' = − 3 5721 × − 1 07 = 3 82   3 8986
V   × l''''' = − 0 6301 × − 1 10 = 0 69   0 2472
7 54   7 5546
= $[pvv]$ = $[pvv]$

The values of $[pvv]$ are found from equations 2, 3, Art III

THIRD SOLUTION—SOLUTION IN TWO GROUPS

The form given in Art. 113 is followed

The Local Equations

(1) At North Base

The Observation Equation

p	(v_1)	(v_2)	
2	+ 1		..
2		+ 1	..
1 +	1	1	+ ;

The Normal Equation

(v_1) (v_2)
16 + 14   19 13 $[p_\epsilon^2]$ upper
14 + 16   19 1. $[pb]$

Solving in general term

(v_1) + 0 50, $[pa]$ 0 13 $[p^1]$
(v_2) 0 13 $[pa^2]$ + 0 50, $[b]$

Hence

(v_1) 0 61
(v_2) 0 61
(v_2) + 0' 61 + 0 61   + 6,
            0 00

and

Local Angle

12 1   00'   40 0,
11 1   40'   04 41
1 12   11'   14 6 9

To find the m s e of a single observation
The value of $[pvv]$ $[pvv]$ 1 7,
Hence for this station, the number of conditions being 4   ,

$$ \mu \quad \sqrt{\frac{1 \cdot 25}{4 - 2}} $$

1".1

(b) At South Base

The Observation Equations

p	(x_4)	(x_5)	l
23	+ 1		0
6		+ 1	0
7	+ 1	+ 1	− 1 07

The Normal Equations

(x_4) (x_5)

$$30 + 7 = -7\ 49$$
$$7 + 13 = -7\ 49$$

Hence

$$(x_4) = -0''\ 13$$
$$(x_5) = -0''\ 50$$
$$(a_0) = -0''\ 13 - 0''\ 50 + 1''\ 07$$
$$= +0''\ 44$$

Local Angles

$$23°\ 08'\ 05''\ 13$$
$$47°\ 31'\ 19''\ 91$$
$$70°\ 39'\ 25''\ 04$$

Also,

$$[pvv] = [px_1] = 3\ 24$$

$$\mu = \sqrt{\frac{3\ 24}{3 - 2}}$$

$$= 1''\ 8$$

The General Adjustment

Most Probable Angles

At N Base, $124°\ 09'\ 40''\ 05 + (1)$
$113°\ 39'\ 01''\ 43 + (2)$
$122°\ 11'\ 15''\ 52 - (1) - (2)$

At S Base, $23°\ 08'\ 05''\ 13 + (4)$
$47°\ 31'\ 19''\ 91 + (5)$
$70°\ 39'\ 25''\ 04 + (4) + (5)$

At Oncota, $34°\ 40'\ 39''\ 66 + (7)$
$43°\ 46'\ 26''\ 40 + (8)$

At Lester, $30°\ 53'\ 30''\ 81 + (9)$

The Angle and Side Equations

(a) Triangle, N Base, S Base, Oneota

Angle SNO 12° 11′ 1, 5° (1) + (1)
    " NSO 23 08′ 05 14 + (1)
    " NOS 34 40′ 39′ 06 + (2)

Sum   160° 00′ 00′ 41
180 + ε   180° 00′ 00 05

         0 0 36 (1) (2) + (3) + 2

(b) Triangle Lester, Oneota, S Base

Angle NSO 70 39′ 25 04 + (4) + (5)
    " SOL 78 27′ 06′ 06 + (5) + (6)
    " OLS 30′ 53′ 30′ 81 + (6)

       180° 00′ 01′ 01
       180 00′ 00″ 17

       0 1′ 14 + (4) + 1 + (5) 1 2 + 2

(c) Quadrilateral N Base, S Base, Oneota, Lester

$$\frac{\sin INS \ \sin LSO \ \sin LON}{\sin LNO \ \sin NSL \ \sin LOS} = 1$$

$LNS =$ 113° 30′ 01″ 13 + (2) LNO 124 05 40 0 + 1
$LSO =$ 70″ 39′ 25″ 04 + (4) + (5) NSL 47 41 04 41 + 63
$LON =$ 43° 16′ 26″ 40 + (8) LOS 58 27 06 06 + 1 + 1

9 9618975,6 − 9,22 (2)   9 9177170,3 14.70 (4)
9 9747660,1 + 7,39½ {(4) + (5)}   9 8677904,5 + 16,2 (5)
9 8399003,4 + 21,98 (8)   9 9911156,1 + 10½ (7) + 1,4

      539,1   509,4
      <u>509,4</u>
      29,7

Check by deducting $\frac{1}{3}$ of the spherical excesses of the triangles from the angles

113° 39′ 04″ 36	124° 09′ 40″ 01
70° 39′ 24″ 92	47° 31′ 19″ 84
43° 46′ 26″ 36	78° 27′ 05″ 93
9 9618976,2	9 9177479,9
9 9747659,3	9 8677848,6
9 8399902,5	9 9911179,8
38,0	8,3
8,3	
29,7	

The two methods agree well

✦ A glance at the log differences for 1″ shows that by expressing them in units of the sixth place of decimals their average value is unity nearly   We have, then, for the side equation,

$$1\ 43\,(1) - 0\ 92\,(2) + 0\ 74\,(4) - 1\ 19\,(5) - 0\ 43\,(7) + 1\ 77\,(8) + 2\ 97 = 0$$

The Weight Equations

$$(1) = -0\ 233\ \overline{|\ 1\ |} + 0\ 267\ \underline{|\ 2\ |}$$

$$(2) = +0\ 267\ |\ 1\ | - 0\ 233\ |\ \underline{2}\ |$$

$$(4) = \qquad\qquad +0\ 038\ |\ \overline{4}\ | - 0\ 021\ \overline{|\ 5\ |}$$

$$(5) = \qquad\qquad -0\ 021\ \overline{|\ 4\ |} + 0\ 088\ |\ 5\ |$$

$$(7) = \qquad\qquad\qquad\qquad\qquad +0\ 032\ |\ \underline{7}\ \overline{|}$$

$$(8) = \qquad\qquad\qquad\qquad\qquad\qquad +1\ 000\ \lfloor\ \bar{8}\ \rceil$$

$$(9) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad +0\ 125\ \overline{|\ \underline{9}\ |}$$

The Correlate Equations

	I	II	III	Check
$\overline{\lfloor\ 1\ \rfloor} =$	− 1		+ 1 43	− 0 43
$\lfloor\ 2\ \rfloor =$	− 1		− 0 92	+ 1 92
$\overline{\lfloor\ 4\ \rfloor} =$	+ 1	+ 1	+ 0 74	− 2 74
$\lfloor\ \bar{5}\ \rfloor =$		+ 1	− 1 19	+ 0 19
$|\ \overline{7}\ | =$	+ 1	+ 1	− 0 43	− 1 57
$\overline{\lfloor\ 8\ \rfloor} =$		+ 1	+ 1 77	− 2 77
$\overline{|\ 9\ |} =$		+ 1		− 1 00

The check is formed by adding each horizontal row (Art 84)

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Expression of the Corrections in Terms of the Correlates

	I	II	III	Check
	+ 0 233		− 0 333	+ 0 100
	− 0 267		− 0 246	+ 0 513
$(1) =$	− 0 034		− 0 579	+ 0 613
	− 0 267		+ 0 382	− 0 115
	+ 0 233		+ 0 214	− 0 447
$(2) =$	− 0 034		+ 0 596	− 0 562
	+ 0 038	+ 0 038	+ 0 028	− 0 104
		− 0 021	+ 0 024	− 0 004
$(4) =$	+ 0 038	+ 0 017	+ 0 052	− 0 108
	− 0 021	− 0 021	− 0 016	+ 0 058
		+ 0 088	− 0 105	+ 0 017
$(5) =$	− 0 021	+ 0 067	− 0 121	+ 0 075
$(7) =$	+ 0 032	+ 0 032	− 0 014	− 0 050
$(8) =$		+ 1	+ 1 770	− 2 770
$(9) =$		+ 0 125		− 0 125

The Corrections in Terms of the Correlates (*collected*)

	I	II	III
$(1) =$	− 0 034		− 0 579
$(2) =$	− 0 034		+ 0 596
$(4) =$	+ 0 038	+ 0 017	+ 0 052
$(5) =$	− 0 021	+ 0 067	− 0 121
$(7) =$	+ 0 032	+ 0 032	− 0 014
$(8) =$		+ 1 000	+ 1 770
$(9) =$		+ 0 125	

Formation of the Normal Equations

	I	II	III	Check
$− (1) =$	+ 0 034		+ 0 579	− 0 613
$− (2) =$	+ 0 034		− 0 596	+ 0 562
$+ (4) =$	+ 0 038	+ 0 017	+ 0 052	− 0 108
$+ (7) =$	+ 0 032	+ 0 032	− 0 014	− 0 050
	[+ 0 138	+ 0 049	+ 0 021	+ 0 200
$(4) =$		+ 0 017	+ 0 052	− 0 108
$(5) =$		+ 0 067	− 0 121	+ 0 075
$(7) =$		+ 0 032	− 0 014	− 0 050
$(8) =$		+ 1	+ 1 770	− 2 770
$(9) =$		+ 0 125		− 0 125
	+ 0 049	+ 1 241	+ 1 687	− 2 978

	I	II	III	Check
+ 1 43(1)				0 852	-- 0 804
0 02(2)				0 533	— 0 564
+ 0 71(3)				0 036	— 0 080
1 19(5)				0 144	- 0 089
0 43(7)				0 006	+ 0 022
+ 1 77(8)				3 133	— 4 903
		0 021		1 057		4 706	- 6 418

The Normal Equations (*collected*)

I	II	III	
+ 0 138	+ 0 049	+ 0 021	0 200
	+ 1 241	+ 1 687	1 540
		+ 4 706	- — 2 970

The solution of these equations gives (page 295)

I	1 507
II	0 642
III	- 0 394

Substitute for I., II , III their values in (4), and we have the general corrections

Adding the local corrections and general corrections together, the total corrections to the measured angles result and are as follows

	Local	General	Total	p	pvv	Final Angles
t_1	0′ 64	0″ 18	0″ 82	2	1 34	124′ 00′ 30″ 87
t_2	0″ 64	+ 0″ 28	0″ 36	2	26	113″ 39′ 1″ 71
t_3	0″ 00	0″ 10	0″ 10	14	50	122′ 11′ 15″ 42
t_4	0′ 13	0″ 09	0″ 22	23	1 10	23′ 08′ 5″ 04
t_5	0″ 50	+ 0″ 04	0″ 46	6	1 27	47′ 31′ 19″ 95
t_6	+ 0″ 44	0″ 05	+ 0″ 39	7	1 06	70′ 39′ 24″ 99
x_7		0″ 07	0″ 07	34	15	34′ 40′ 39″ 59
t_8		1″ 33	1″ 33	1	1.77	43′ 46′ 25″ 07
t_9		0″ 08	- 0″ 08	8	05	30′ 53′ 30″ 73
				$\lvert pvv \rvert$	7 50	

Number of local conditions	2
Number of general conditions	3
Total	5

The method of solution just given is substantially the same as that employed on the survey of the Great Lakes between Canada and the United States by the U. S Engineers

The Precision of the Adjusted Value

(a) To find the m s e of an observation of weight unity

Computation of $[\rho vv]$

(1) From the preceding table $[\rho vv]$ has been found directly, thus

$$[\rho vv] \quad 7\ 50$$

(2) Check (Art 113) From the station adjustments find $[\ ]$

N Base gives (p 288) 1 75
S Base gives (p 289) 3 24

$$3\ 99 \quad |\ \cdot\ |$$

From the general adjustment find $[wuv]$.

$$(\alpha)\ l_0' \times \text{I} \quad -0\ 26\ \times\quad 1\ 50\ \quad 1\ 0\ 1^*$$
$$l_0'' \times \text{II} \quad 1\ 54\ \times\quad 0\ 642\quad 1\ 0\ 0\ 9\ 4$$
$$l_0''' \times \text{III} \quad -2\ 07\ \times\quad 0\ 304\quad 1\ 4\ 16$$

$$1\ 2\ 47$$

$$(\beta)\ l_0' \times \dfrac{l_0'}{[\ a\text{A}\ ]} \quad 0\ 26\ \times\quad 1\ 884\quad 1\ 0\ 49$$

$$l_0''\ 1 \times \dfrac{l_0''\ 1}{[\ \delta\text{B}\ 1\ ]} \quad 1\ 45\ \times\quad 1\ 154\quad 1\ 1\ 7$$

$$l_0'''\ 2 \times \dfrac{l_0'''\ 2}{[\ cc\ 2\ ]} \quad 0\ 04\ \times\quad 0\ 304\quad 1\ 0\ 47$$

$$1\ 2\ 0$$

$$[\ vuv\ ]\quad 2\ 5\ 7$$

and $[\rho vv]\quad 1\ 99\ 1\ 2\ 58$

$$7\ 57$$

Hence taking the mean of the values of $[\rho vv]$,

$$\mu \quad \sqrt{\dfrac{7\ 54}{2\ 1\ 3}} \quad 1^*\ 24$$

there being 2 local conditions and 3 net conditions.

(b) To find the m s e of an angle in the adjusted figure

Angle $N\ l\ S$

dF (2) (5)

$\downarrow\ 0.055\ 1 \quad 0\ 067\ \text{II}\ \downarrow\ 0\ 308\ \text{III}$

from the weight equations

From equations 25, Art 114,

$$q_1\ -[\alpha\alpha]g_1 + [\alpha\beta]g_2 +$$
$$q_2\quad [\alpha\beta]g_1 + [\beta\beta]g_2 +$$
$$\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot$$

The values of $[aa]$, $[ab]$ ... are given in the weight equations. Hence

$$y_1 \quad + 0.767 \quad a \quad - 0.733 \, e \quad + 1 \quad + 0.233$$
$$y_2 \quad 0.733 \times + 0 + 0.267 \, e \quad + 1 \quad - 0.267$$
$$y_3 \quad + 0.0.98 \quad a \quad - 0.021 \quad + 1 \quad + 0.021$$
$$y_4 \quad 0.021 \quad a + 0.085 \, e \quad + 1 \quad - 0.085$$

		y		y		$\begin{vmatrix} a \end{vmatrix}$	0.022	(See the solution of the normal equations.)
						$\begin{vmatrix} ab \end{vmatrix}$	0.006	
0	+ 0.733	0		$\begin{vmatrix} ab \end{vmatrix}$			
1	0.767	0.267					
0	0.021	0		$\begin{vmatrix} ac \end{vmatrix}$	0.271		
1	0.085	0.085		$\begin{vmatrix} ab \end{vmatrix}$			
	$[y]$	0.138			0.303		

$$u_1 \quad + 0.003$$
$$u_b \quad + 0.23 + 0.003$$
$$0.38$$

(c) To find the m s e of a side in the adjusted figure.

Side   Oneota Leater

$$v \quad OI \quad NS \frac{\sin ONS \, \sin OSL}{\sin SOV \, \sin OLS}$$

Therefore

$$dl \quad + 1.0(I) + 1.0(9) + 0.74(B) + 0.74(S) \quad +0.1(7) \quad +1.52(9)$$

in units of the sixth place of decimals,

$$0.17_3 \, I \quad 0.1/5 \, II \quad + 0.015 \, III.$$

from the weight equation.

The solution is carried through exactly as in the preceding case. We find

$$[gy] \quad 2.011 \text{ and } u_l \quad + 1.9$$

Hence $u + u_b \quad + 1.9 + 1.9$

+ 5, in units of the sixth place of decimals.

Now $\log OI \quad 4.21$ ... $OI \quad 16.560^m$

m s e of side $\dfrac{16.560}{0.443} \approx 0.000014$

$0^m.06$

Solution of the Normal Equations

I	II	III	l	f (angle)
+ 0 138	+ 0 049	+ 0 021	— 0 260	+ 0 055
	+ 1 241	+ 1 687	— 1 540	— 0 067
		+ 4 706	— 2 970	+ 0 700
+ 1	+ 0 355	+ 0 152	— 1 885	+ 0 399
	+ 1 224	+ 1 680	— 1 446	— 0 067
		+ 4 703	— 2 930	+ 0 692
				+ 0 022
	+ 1	+ 1 373	— 1 183	— 0 071
		+ 2 396	— 0 945	+ 0 811
				+ 0 006
		+ 1	— 0 394	+ 0 338
				+ 0 274

Ex 1 Adjust the observed differences of longitude* given in the following table
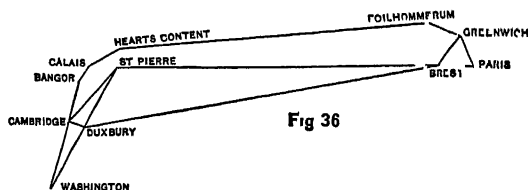
Fig 36

Dates		Observed Differences			Corrections
		h	m s	s	
1851	Cambridge–Bangor,	0	9 23 080 $\pm$ 0 043		v_1
1857	Bangor–Calais,	6	00 316 $\pm$ 0 015		v_2
1866	Calais–Heart's Content,	55	37 973 $\pm$ 0 066		v_3
1866	Heart's Content–Foilhommerum,	2	51 56 356 $\pm$ 0 029		v_4
1866	Foilhommerum–Greenwich,	41	33 336 $\pm$ 0 049		v_5
1872	Brest–Greenwich,	17	57 598 $\pm$ 0 022		v_6
1872	Brest–Paris,	27	18 512 $\pm$ 0 027		v_7
1872	Greenwich–Paris,	9	21 000 $\pm$ 0 038		v_8
1872	St Pierre–Brest,	3	26 44 810 $\pm$ 0 027		v_9
1872	Cambridge–St Pierre,	59	48 608 $\pm$ 0 021		v_{10}
1869–1870	Cambridge–Duxbury,	1	50 191 $\pm$ 0 022		v_{11}
1870	Duxbury–Brest,	4	24 43 276 $\pm$ 0 047		v_{12}
1867 / 1872	Washington–Cambridge,	23	41 041 $\pm$ 0 018		v_{13}
1872	Washington–St Pierre,	1	23 29 553 $\pm$ 0 027		v_{14}

* *Coast and Geodetic Survey Report*, 1880, app No 6

[Number of conditions $= n - s + 1$, where n is number of observed differences of longitude, and s is number of longitude stations

The condition equations are

$$-v_6 + v_7 - v_8 = + 0.086$$
$$-v_1 - v_2 - v_3 - v_4 - v_5 + v_6 + v_9 + v_{10} = + 0.045$$
$$-v_9 - v_{10} + v_{11} + v_{12} = -0.049$$
$$+ v_{10} + v_{13} - v_{14} = -0.006$$

The weights are taken inversely as the squares of the p   e
Solution by method of correlates, as in Art 110]

Ex 2 The system of triangulation shown in the figure was executed by Koppe in the determination of the axis (Airolo - Goschenen) of the St Gothard tunnel * In the following table the adjusted values are given side by side with the measured values It is proposed as a problem of adjustment

Fig 37
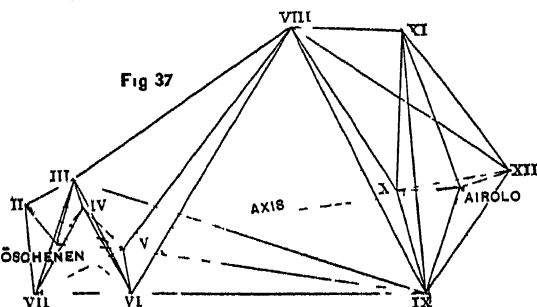
At Goschenen	Measured				Adjusted	
II	0	00'	00" 00	00" 00		
III	44	33'	10" 88	10" 03		
IV	69	30'	12" 51	11" 62		
V	124	58'	1" 23	5" 13		
At II						
III	0'	00'	00" 00	00" 00		
IV	37"	53'	51" 33	52" 07		
V	60"	29'	33" 13	33" 82		
VI	77'	4'	5" 67	8" 17		
Goschenen	93'	11'	41" 69	40" 57		
VII	124	16'	33" 98	33" 27		
At III						
VIII	0	00'	00" 00	00" 00		
IX	53"	58'	14" 48	15" 49		
VI	99	47'	50" 21	50" 86		
IV	102'	32'	51" 36	51" 90		
Goschenen	138	44'	28" 81	29" 70		
VII	144"	28'	12" 47	11" 40		
II	180'	50'	38" 94	39" 11		

At IV	Measured				Adjusted	
V	0'	00'	00" 00	00" 00		
VI	15"	41'	3" 57	6" 29		
VII	74	12'	20" 55	19" 86		
Goschenen	80 '	32'	48" 90	50" 12		
II	135"	44'	49" 77	50" 91		
III	199"	24'	11" 56	10" 73		
At V						
IV	0"	00'	00" 00	00" 00		
VIII	78"	40'	5" 91	6" 72		
IX	140"	44'	43" 51	44" 45		
VI	215"	32'	45" 41	43" 45		
VII	286	19'	25" 30	27" 21		
Goschenen	316"	00'	44" 92	43" 61		
II	338"	20'	33" 53	31" 74		
At VII						
II	0"	00'	00" 00	00" 00		
III	19"	11'	58" 44	59" 03		
IV	32"	4'	49" 32	48" 68		
V	61"	11'	54" 08	56" 05		
VI	90"	05'	39" 47	37" 00		

At VIII

XI	0°	00'	00" 00	00" 00
XII	18°	56'	17" 43	17" 54
X	43°	50'	24" 03	24" 70
IX	50°	18'	22" 52	20" 27
VI	106°	30'	15" 04	15" 37
V	112°		28" 72	29" 24
III	130°	11'	30" 81	41" 54

At IX

VI	0°	00'	00" 00	00" 00
V	8°	28'	17" 13	15" 06
III	18°	33	3" 27	5" 00
VIII	63°	41'	28" 63	28" 55
X	76°	59'	50" 89	51° 48
XI	79°	10'	36" 33	36" 34
Airolo	109°	45'	39" 23	39" 33
XII	123°	16'	23" 76	24" 23

At X

XII	0°	00'	00" 00	00" 00
Airolo	9°	49'	30" 02	37" 92
IX	91°	30'	5" 16	5" 96
VIII	252°	43'	46" 75	47" 49
XI	275°	12'	8" 44	9" 74

At XI

XII	0°	00'	00" 00	00" 00
Airolo	16°	55'	55" 06	54" 38
IX	37°	13'	59" 79	58" 13
VIII	152°	26'	30" 24	30" 44

At XII

IX	0°	00'	00" 00	00" 00
Airolo	30°	31'	2" 30	3" 30
X	42°	13'	20" 53	21" 33
VIII	90°	3'	2" 22	1" 74
XI	98°	40'	14" 95	13" 72

At Airolo

XI	0°	00'	00" 00	00" 00
XII	94°	54'	50" 06	55" 26
IX	230°	53'	7" 51	6" 98
X	296°	26'	40" 43	51" 11

The distance X–XII is 4416^m 8
There are 19 angle equations and 15 side equations in the adjustment

SOLUTION BY SUCCESSIVE APPROXIMATION

141 The rigorous forms of solution which have been given are suitable for a primary triangulation where the greatest precision is required   In secondary or tertiary work it would not be advisable to spend so much labor in the reduction, since the systematic errors remaining would probably largely outweigh the accidental errors eliminated. For work of this kind the method of solution by successive approximation is to be preferred

The principle underlying the process of solution is that explained in Art 115   Each condition or set of conditions

is adjusted for independently in succession, the values of the corrections found at each adjustment being closer and closer approximations to the final values. Should the values found after going through all of the conditions not satisfy the first, second, ... groups of condition equations closely enough, the process must be repeated till the required accuracy is attained.

To make the operation as simple as possible let us take but a single condition at a time.

(1) Local equation at N Base,

$$c_5 + c_6 + c_7 + 1.37 = 0$$

The solution is given in Ex. 1, Art. 111,

$$c_5 = 0''.64, \quad c_6 = 0''.64, \quad c_7 = 0''.0x$$

(2) Local equation at S Base,

$$c_5 + c_6 + c_9 + 1.07 = 0$$

The solution is given in Ex. 2, Art. 111,

$$c_5 = 0''.13, \quad c_6 = 0''.50, \quad c_9 = +0''.44$$

(3) Angle equation,

$$c_5 + c_6 + c_7 + 0.48 = 0$$

Using the values of c_5, v_6 already found as first approximations, the equation reduces to

$$c_5 + c_6 + c_7 + 0.96 = 0$$

The method of solution is given in Ex. 2, Art. 110,

$$c_5 = 0''.13, \quad v_6 = 0''.08, \quad v = 0''.05$$

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The successive approximations found for them when added, give

v_1 0.64
c_1 $0''.64$
v_1 $0'' .$
c_1 $0''.21$

Proceed similarly with the remaining two condition equations. The resulting values will agree closely with the rigorous values already found.

142. In order to bring out still more clearly the advantages of solving in this way, let us take a more extended example. A good one is furnished by the triangulation (1874‑1878) of the east end of Lake Ontario, omitting the system around the Sandy Creek base.
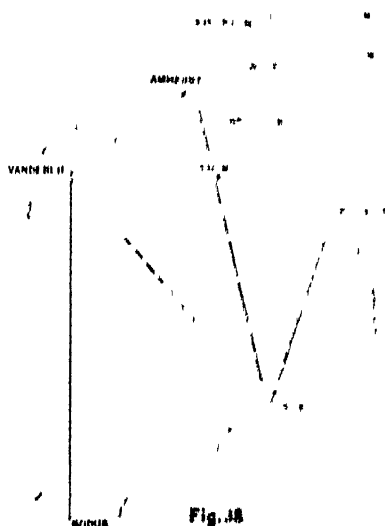
Fig. 48

The measured values of the angles are given in the following table. Each angle is taken to be of the same weight. In the last column are given the locally corrected angles found by the rigorous methods of solution.

Station occupied	Angle to be measured between				Locally corr. angles
Sir John,	Carlton and Kingston,	90	17′	11″ 91	
	Wolfe and Kingston,	56	·1′	00′ 77	
Carlton,	Wolfe and Sir John,	120	15′	06″ 51	
	Kingston and Sir John	62	03′	·7 56	
Kingston,	Sir John and Wolfe,	64	16′	50 91	
	Carlton and Wolfe,	37	02′	04 13	
	Wolfe and Amherst,	86	10′	14′ 70	
Wolfe,	Duck and Carlton,	155	07′	16 51	
	Amherst and Carlton,	140	1·′	11′ 11	
	Kingston and Carlton,	81	13	11′ 11	
	Sir John and Carlton,	25	15′	16′ 80	
Amherst	Kingston and Wolfe,	35	11′	23″ 00	23′ 00
	Kingston and Duck,	111	15′	28′ 16	28″ 65
	Wolfe and Duck,	76	04′	06′ 1·	06″ 00
	Grenadier and Duck,	51	15′	00′ 11	
	Duck and Vanderlip,	71	1·′	55′ 11	05′ 32
	Vanderlip and Kingston,	1·6	50′	06′ 11	06′ 00
Duck	Oswego and Vanderlip,	104	05	58′ 93	50 10
	Vanderlip and Amherst,	70	·6′	11′ 99	1· 16
	Amherst and Wolfe,	56	01′	1· ·17	1· 61
	Wolfe and Grenadier	15	16′	13 16	11′ 53
	Grenadier and Stony Pt	19	53′	1·′ 77	12′ 91
	Stony Pt and Oswego,	60	11′	19′ 16	19 63
Grenadier,	Stony Pt and Duck,	7··	11′	11 01	11′ 81
	Duck and Amherst,	56	15′	01″ 18	01′ 19
	Duck and Stony Pt,	251	16′	28′ 89	·6 16
	Amherst and Stony Pt,	·31	11′	2·′ 01	21′ 07
Stony Pt	Oswego and Duck,	88	22′	00′ 86	
	Duck and Grenadier,	51	53′	12′ 60	12 70
	Grenadier and Duck,	408	06′	17″ 21	17″ 30
Oswego,	Sodus and Vanderlip,	80	26′	16″ 10	16′ 50
	Sodus and Duck,	107	19′	03′ 28	03 90
	Sodus and Stony Pt,	11·	12′	10′ 11	15 28
	Vanderlip and Duck,	26	10′	16′ 61	17′ 17
	Vanderlip and Stony Pt,	87	13′	01′06	01′ 09
	Duck and Stony Pt,	10	53	12′58	11 1·
Vanderlip,	Amherst and Duck,	55	15′	07″ 12	07′ 10
	Amherst and Oswego,	5·7	19′	53′ 17	53″ 16
	Duck and Oswego,	10	01′	15′ 11	15′ 86
	Duck and Sodus,	5·7	50′	12′55	1·″ 12
	Oswego and Sodus,	35	5·′	26′ 55	26′ 56
	Sodus and Amherst,	·11	1·′	10′ 11	10′ 28
Sodus,	Vanderlip and Oswego,	60	12′	5·″ 55	

The local and general equations aɪe foɪmed as usual (see Aɪts 117-140)   The general ɪule ɪn the solutɪon ɪs to adjust for one condɪtɪon at a tɪme   Instead, howevcɪ, ot foll0wɪng out thɪs rule stɪɪctly, ɪt ɪs often better to adjust foɪ a *group* of condɪtɪons sɪmultaneously   Often a gɪoup ɪs almost as easɪly managed as a sɪngle condɪtɪon   No ɪule can be gɪven to cover all cases, and much must be left to the judgment and ɪngenuɪty of the computer

143 The Local Adjustment at Each Station.—
(a) Adjust for each sum angle sepaɪately
     Rule and example ɪn Aɪt 121
(b) Adjust for closure of the hoɪɪzon
     Rule and example ɪn Aɪt 121

At statɪons Sɪr John, Carlton, Kɪngston, Wolfe theɪe aɪe no local condɪtɪons, and at each of the statɪons Amheɪst, Stony Poɪnt, Sodus theɪe ɪs one angle ɪndependent of the others, and therefore not locally adjusted

The angles at statɪon Amherst may be rɪgorously ad-justed, as ɪn Art 121   The resultɪng values aɪe gɪven ɪn the table   If we break the adjustment ɪnto two paɪts, as ɪn (a) and (b), we have

(a) *Sum Angle*

	Measured values			Adjusted
Kɪngston–Wolfe,	35°	41	23″ 02 − 0″ 29	22″ 73
Wolfe–Duck,	76°	04′	06″ 32 − 0′ 29	06″ 03
	111°	45′	29″ 34	28″ 76
Kɪngston–Duck,	111°	45′	28″ 46 + 0″ 29	28″ 75 check
			3)0″ 88	
			0″ 29	

(b) *Closure of Horɪzon*

Kɪngston–Wolfe,	35°	41′	22″ 73 − 0″ 08	22″ 65
Wolfe–Duck,	76°	04′	06″ 03 − 0″ 07	05″ 96
Duck–Vanderlɪp,	71°	15′	25″ 43 − 0″ 08	25″ 35
Vanderlɪp–Kɪngston,	176°	59′	06″ 11 − 0″ 07	06″ 04
			4)00″ 30	00″ 00 check
			00″ 075	

The adjusted values agree closely wɪth those from the sɪmultɪneous solutɪon, as gɪven ɪn the table

•

At station Duck the angles close the horizon   Hence the correction to each angle is one-sixth of the difference of their sum from 360°   (See Art 121)

144 **The General Adjustment.**—The local adjustment being finished, we shall consider the adjusted angles to be independent of one another and to be of the same weight   We are therefore at liberty to break up the net into its simplest parts   We have in our figure, first a quadrilateral *SCWK*, next two single triangles *KWA*, *AWD*, next a central polygon *DAGSOV*, and, lastly, a single triangle *VOS*   These three figures include most cases that arise in any triangulation net

(a 1) *Adjustment of a Quadrilateral* —In the quadrilateral *SCKW* all of the eight angles 1, 2,   8 are supposed to be equally well measured

(1) The Angle Equations

The angle equations from the triangles *SCW*, *CWK*, *WKS* may be written in general terms

Fig 39

$$v_1 + v_2 + v_3 + v_4 = l_1$$
$$v_1 + v_1 + v_5 + v_6 = l_2$$
$$v_5 + v_6 + v_7 + v_8 = l_3$$
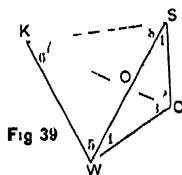
As these equations are entangled, if we adjusted for each in succession a great many repetitions of the adjustment would be necessary to obtain values that would satisfy the equations simultaneously   It is, therefore, better to adjust simultaneously, and it happens that a very simple rule for doing this can be found

Call k_1, k_2, k_3, k_4 the correlates of the equations in order , then the correlate equations are

$$
\begin{aligned}
k_1 &= v_1 & k_2 + k_3 &= v_5 \\
k_1 &= v_2 & k_2 + k_3 &= v_6 \\
k_1 + k_2 &= v_3 & k_3 &= v_7 \\
k_1 + k_2 &= v_4 & k_3 &= v_8
\end{aligned}
$$

and the normal equations

$$4k_1 + 2k_2 \qquad = l_1$$
$$2k_1 + 4k_2 + 2k_3 = l_2$$
$$\qquad 2k_2 + 4k_3 = l_3$$

Solving these equations, there result

$$k_1 = \tfrac{1}{8}(+\,3l_1 - 2l_2 +\; l_3)$$
$$k_2 = \tfrac{1}{8}(-\,2l_1 + 4l_2 - 2l_3)$$
$$k_3 = \tfrac{1}{8}(+\; l_1 - 2l_2 + 3l_3)$$

Substitute these values in the correlate equations, and

$$v_1 = v_2 = \tfrac{1}{8}(+\,3l_1 - 2l_2 +\; l_3)$$
$$v_3 = v_4 = \tfrac{1}{8}(+\; l_1 + 2l_2 -\; l_3)$$
$$v_5 = v_6 = \tfrac{1}{8}(-\; l_1 + 2l_2 +\; l_3)$$
$$v_7 = v_8 = \tfrac{1}{8}(+\; l_1 - 2l_2 + 3l_3)$$

which may be written

$$v_1 = v_2 = \tfrac{1}{4}l_1 - \tfrac{1}{4}(l_2 - \tfrac{1}{2}l_3 - \tfrac{1}{2}l_1)$$
$$v_3 = v_4 = \tfrac{1}{4}l_1 + \tfrac{1}{4}(l_2 - \tfrac{1}{2}l_3 - \tfrac{1}{2}l_1)$$
$$v_5 = v_6 = \tfrac{1}{4}l_3 + \tfrac{1}{4}(l_2 - \tfrac{1}{2}l_3 - \tfrac{1}{2}l_1)$$
$$v_7 = v_8 = \tfrac{1}{4}l_3 - \tfrac{1}{4}(l_2 - \tfrac{1}{2}l_3 - \tfrac{1}{2}l_1)$$

whence follows at once the convenient rule for adjusting the quadrilateral, so far as the angle equations are concerned

(α) *Write the measured angles in order of azimuth in two sets of four each, the first set being the angles of SCW, and the second those of WKS*

(β) *Adjust the angles of each set by one-fourth of the difference of this sum from* 180° + *excess of triangle, arranging the adjusted angles in two columns, so that the first column will show the angles of SCK, and the second those of CWK*

(γ) *Adjust the first column by one-fourth of the difference of its sum from* 180° + *excess of triangle, and apply the same correction, with the sign changed, to the second column*

The spherical excesses of the triangles SCW, CWK, WKS being $0''$ 16, $0''$ 35, and $0''$ 47 respectively, the adjustment of the quadrilateral may be arranged as follows

Measured angles			Adjusted angles
33 53′ 35″ 14	35″ 56		35″ 40
62′ 03′ 27″ 56	27″ 98		27″ 82
58 44′ 38″ 96		39″ 40	39″ 56
25′ 18′ 16″ 80		17″ 22	17″ 38
179′ 59′ 58″ 48			00″ 16 check
180 + ε = 180° 00′ 00″ 16			
4)1″ 68			
0″ 42			
58° 54′ 57″ 54		58″ 11	58″ 27
37 02′ 04″ 43		04″ 99	05″ 15
27′ 38′ 46″ 48	47″ 04		46″ 88
56′ 24′ 9″ 77	10″ 33		10″ 17
179° 59′ 58″ 22	00″ 91		00″ 47 check
180 + ε = 180 00′ 00″ 47	00″ 28		
4)2″ 25	4)0″ 63		
0″ 56	0″ 16		

(2) The Side Equation

Using the values of the angles just found, we next form the side equation with pole at O It is

$$\frac{\sin OSC}{\sin SCO} \frac{\sin OCW}{\sin CWO} \frac{\sin OWK}{\sin WKO} \frac{\sin OKS}{\sin KSO} = 1$$

or writing it in general terms when reduced to the linear form (see Art 129)

$$a_1 v_1' + a_2 v_2' + a_3 v_3' + a_4 v_4' + a_5 v_5' + a_6 v_6' + a_7 v_7' + a_8 v_8' = l_4$$

where v_1', v_2',   are the corrections resulting from the side equation

Solving as in Ex 2, Art 110, we have the corrections

$$v_1' = \frac{a_1}{[aa]} l_4, \; v_2' = \frac{a_2}{[aa]} l_4,$$

These corrections may be found still more rapidly as follows Since the side equation may be so transformed that the coefficients a_1, a_2,   are approximately equal to unity numerically (see Art 131), we may take each of them to be unity, and then

$$v_1' = v_3' = v_5' = v_7' = + \tfrac{1}{8} l_1$$
$$v_2' = v_4' = v_6' = v_8' = - \tfrac{1}{8} l_1$$

that is, *the corrections to the angles are numerically equal, but are alternately + and −*

This plan has the additional advantage of not disturbing the angle equations

Returning to our numerical example, we first reduce the side equation to the linear form

OSC	$= 33°\ 53'\ 35''\ 40 + v_1$	SCO	$= 62°\ 03'\ 27''\ 82 + v_2$
OCW	$= 58°\ 44'\ 39''\ 56 + v_3$	CWO	$= 25°\ 18'\ 17''\ 38 + v_4$
OWK	$= 58°\ 54'\ 58''\ 27 + v_5$	WKO	$= 37°\ 02'\ 05''\ 15 + v_6$
OKS	$= 27°\ 38'\ 46''\ 88 + v_7$	KSO	$= 56°\ 24'\ 10''\ 17 + v_8$

$9\ 7463587 + 31\ 3v_1$	$9\ 9461673 + 11\ 2v_2$
$9\ 9318952 + 12\ 8v_3$	$9\ 6308691 + 44\ 5v_4$
$9\ 9326832 + 12\ 7v_5$	$9\ 7797125 + 27\ 9v_6$
$9\ 6665301 + 40\ 2v_7$	$9\ 9206181 + 14\ 0v_8$
72	70
70	
$\overline{2}$	

Dividing by 20, which will reduce the coefficients to unity approximately, and

$$1\ 56v_1' - 0\ 56v_2' + 0\ 64v_3' - 2\ 22v_4' + 0\ 64v_5'$$
$$- 1\ 40v_6' + 2\ 01v_7' - 0\ 70v_8' + 0\ 10 = 0$$

Hence

$$[aa] = 15$$

and

$$v_1' = -0''\ 01,\ v_2' = 0''\ 00,\ v_3' = 0''\ 00,\ v_4' = +0''\ 01,\ \text{etc}$$

By the second rule the corrections would be $\mp \dfrac{0''\ 1}{8}$, that is, $\mp 0''\ 01$ alternately, which values differ but little from the preceding

The total corrections to the angles are the sums of the two sets of corrections from the angle and side equations

(a 2) *Adjustment of a Quadrilateral* —By the following artifice the quadrilateral may be *rigorously* adjusted for the side equation without disturbing the angle equation adjustment, which amounts to the same thing as the simultaneous adjustment of the angle and side equations

Suppose that the angle equations have been adjusted as already explained in (a 1) If v_1', v_2', v_8' denote the corrections arising from the side equation, the condition equations may be written

$$v_1' + v_2' + v_3' + v_4' = 0$$
$$v_4' + v_3' + v_5' + v_6' = 0$$
$$v_6' + v_5' + v_7' + v_8' = 0$$
$$a_1v_1' + a_2v_2' + a_3v_3' + a_4v_4' + a_5v_5' + a_6v_6' + a_7v_7' + a_8v_8' = l_1'$$

By writing the corrections in the form

$v_1' = + v + v'$	$v_5' = + v + v'''$
$v_2' = + v - v'$	$v_6' = + v - v'''$
$v_3' = - v + v''$	$v_7' = - v + v''''$
$v_4' = - v - v''$	$v_8' = - v - v''''$

the first three condition equations become $0 = 0$ identically, and we have therefore to deal only with the single condition equation

$$(a_1 + a_2 + a_5 + a_6 - a_3 - a_4 - a_7 - a_8)v + (a_1 - a_2)v'$$
$$+ (a_3 - a_4)v'' + (a_5 - a_6)v''' + (a_7 - a_8)v'''' = l_1'$$

with

$$(v + v')^2 + (v - v')^2 + (-v + v'')^2 + (-v - v'')^2 +   = \text{a min}$$

The correlate equations are

$$(a_1 + a_2 + a_5 + a_6 - a_3 - a_4 - a_7 - a_8)\lambda = 4v$$
$$(a_1 - a_2)\lambda = v'$$
$$(a_3 - a_4)\lambda = v''$$
$$(a_5 - a_6)k = v'''$$
$$(a_7 - a_8)k = v''''$$

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Substitute in the condition equation, and

$$k\{\tfrac{1}{4}(a_1 + a_2 + a_3 + a_4 - a_5 - a_6 - a_7 - a_8)' + (a_1 \quad a_2)'$$
$$+ (a_3 - a_4) + (a_5 - a_6)^2 + (a_7 \quad a_8)'\} \quad l_1'$$

from which k can be found

Hence the corrections are known

The complete adjustment of our quadrilateral is contained in the following table

Meas Angles		Local Angles	Log Sines	Log Diff	$\frac{v}{v''}$	$\frac{v}{m}$
33° 53′ 35″ 14	35″ 56	35″ 40	9 7463567	31 3	1 5	1606
62° 03′ 27″ 56	27″ 98	27″ 62	9 9101073	11 2		
58° 44′ 38″ 98	39″ 40	39″ 56	9 9318952	1 8	57 3	3 53
25° 18′ 16″ 80	17′ 22	17″ 38	9 6308691	41 5		
58″ 48						
00″ 16						
1″ 68						
58° 54′ 57″ 54	58″ 11	58″ 27	9 9376832	1 7	40 6	1648
37° 02′ 04″ 43	04″ 90	05″ 15	9 7708125	7 0		
27° 38′ 46″ 48	47″ 04	46″ 88	9 6665301	40	54	9094
56° 24′ 09″ 77	10″ 33	10″ 17	9 9206181	11 0		
58″ 22	00″ 91		72	70	10 5	9 1
00″ 47	00″ 28		70		9 1	
2″ 25	0″ 63		2		4)10 4	
					2 6	97
						970

$$\frac{v}{2\ 6} = \frac{v'}{42\ 5} = \frac{v''}{57\ 3} = \frac{v'''}{40\ 6} = \frac{v''''}{54\ 2} = \frac{2}{9702}$$

Hence the corrections are known   These corrections, applied to the local angles, give the final angles required

(b) *Adjustment of a Single Triangle*

Rule and example in Ex 2, Art 109, and Ex 2, Art 110

The single triangles in our figure are Kingston, Wolfe, Amherst, Wolfe, Duck, Amherst, Vanderlip, Oswego, Sodus

For example, take the first ($t = 0'' 72$)

		Measured		Adjusted
Kingston,	55	$19'$	$14'' 70$	$15'' 78$
Wolfe,	55	$59'$	$20'' 10$	$21'' 18$
Amherst,	35	$41'$	$22'' 00$	$23'' 77$
	179	$59'$	$57' 49$	$0'' 73$ check
$180 + t = 180$		$00'$	$00'' 72$	
			$3)3'' 23$	
			$1'' 08$	

(c) *Adjustment of a Central Polygon* —In the central polygon Duck, Amherst, Grenadier, Stony Point, Oswego, Vanderlip the condition equations in general terms are

Fig 40
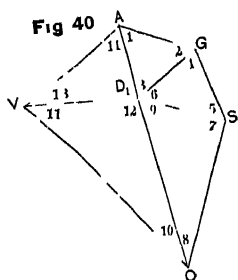
Local equation (horizon equation),

$$v'_1 + v'_6 + v'_9 + v'_{12} + v'_{15} = l_1$$

Angle equations,

$$v'_1 + v'_2 + v'_3 = l_2$$

$$v'_4 + v'_5 + v'_6 = l_3$$

$$v'_{13} + v'_{14} + v'_{15} = l_6$$

Side equation (pole at Duck),

$$a_1 v'_1 + a_2 v'_2 + a_3 v'_3 + a_5 v'_5 + \quad + a_{11} v'_{14} = l$$

We may adjust for these equations in order, first the horizon equation, then the angle equations separately, as they are not entangled, and next the side equation

A rigorous adjustment may, however, be carried out at once with very little additional labor Adjust first each angle equation by itself, and let (v'_1), (v'_2), be the values that result Let (1), (2), denote the farther corrections to the measured angles in order arising from the local and side equations, so that

$$v'_1 = (v'_1) + (1)$$

$$v'_2 = (v'_2) + (2)$$

310   THE ADJUSTMENT ...

If we substitute these values in ... we have the new condition equations

$$a(1) + a(2) \quad \dots$$
$$(3) + (6) + \dots$$
$$(1) \dots$$

from which to find (5, (6, (7, ...

Calling $k_1, k_2, l, H,$ ... the ... equations in order, we have the ...

(1) $a_1 k_1 + l$
(2) $a_2 k_2 + l$
(3) $k_3 + l$

Eliminate now the angle equation ...

$$a_1 + a l_1^2 + \dots$$
$$a_2 + a l_2^2 + \dots$$

Hence

(1)   ...
(2)   ...
(3)   ...

Substitute these values of ... equations, and we have the normal ...

$$2[aa] \quad a_1 a_1 \quad a_2 a$$
$$[a l]$$

Solving, we find k_1, k_2 and thence ...

The normal equations may be written down directly in every case, as the law of their formation is as evident as that of ordinary normal equations. They involve only two unknowns, l, ll no matter how many sides the polygon has.

It is evident that the elimination of the angle equation correlates l, ll, from the correlate equations before forming the normal equations amounts to the same thing as first forming the normal equations in the usual way and then eliminating l, ll, For the normal equations from the correlate equations are

$$
\begin{array}{cccccc}
\text{and } z, & | (a_1 + a_2)l | (a_3 + a_4) \, ll \, | & & l' \\
& | \, z_1 & | \, l & | \, ll \, | & & l'' \\
(a_1 + a_2)z_2 & | \, l_2 & | \, l & & & 0 \\
(a_3 + a_4)z_3 & | \, l_3 & & | \, 3ll & & 0
\end{array}
$$

Substitute for l, ll, their values from the third, fourth, ... equations in the first two, and we find the two normal equations as before.

The introduction of the method of eliminating the correlates arising from one set of condition equations before forming the normal equations is due to Schleiermacher, of the Hessian survey. For a fuller account of it see Fischer's *Geodasie*, part III p. 93, Hügel in *General Bericht der europäischen Gradmessung*, 1864, pp. 100 seq., Nell in *Zeitschrift für Vermess.*, vol. X pp. 1 seq., vol. XII pp. 313 seq.

The process is not of any special advantage except in such problems as that under discussion, and then it is better to use the final formula for the normal equations directly.

We shall now proceed with our numerical example.

At station Duck the measured values of the angles are taken, at the other stations the locally adjusted angles

Given Angles		Log Sines	Diff 1″	Square	Products	Sums
53° 38′ 00″ 35	00″ 6	9 911 900				
50° 35′ 01″ 10	01″ 47		9 56 701 56	11 9 / 17 3	0 / 91 3	67	4
71° 40′ 55″ 05	56″ 1					
180° 00′ 00″ 40						
101° 00′ 01″ 1	180	4				
30″ 65						
0″ 48						
75° 13′ 33″ 84	33″ 40	9 990 704				
51° 54′ 1 ″ 70	1 ″ 35		9 49 060	14 4 / 10 5	11 4 / 7	7 6	11 1
19° 53′ 1 ″ 71	14″ 4					
59″ 31						
1″ 6						
1″ 95						
0″ 65						
46° ° ′ 00″ 16	00″ 47	9 999° 55				33 6
30° 54′ 11″ 3	11″ 03		9 71 530	1 5	1 9 4	1 1	
00° 41′ 10″ 40	10″ 47					
01″ 64						
3″ 17						
1 ′ 17						
0″ 39						
0° 19′ 17″ 17	15″ 15	9 643 4				
39° 01′ 15″ 86	16″ 61		9 57797 40	11 6 / 1 3	17 30 6 / 14 9	701 3	4
104° 06′ 55″ 95	56″ 70					
′ 10						
3″ 49						
0″ 40						
0″ 76						
39° 18′ 02″ 30	00″ 4	9 79° 537				4 170
71° 11′ 15″ 3	1 ″ 46		9 970 454	0 7 / 7 5	71 9 / 51 4	160 4	4
70° ° 6′ 31″ 99	31″ 6					
01″ 61	64 3	0 47	4 50 4 / 100 4	1 54 4	4	
1″ 71	0 47					
5″ 00	44	0	04 5 5			
0″ 97	4					
744		1 367 6				

The Normal Equations   Local Equation at Station Duck

12 307k_1 + 6 24k_2 24 3   74 16′ 56 12
 6 24k_1 + 10k_2 2 01   49 54′ 14 4
                                            00 44′ 19 07
k_1 0 020   104 05′ 50 40
k_2 + 0 21 3   70 36′ 41 02

                                            359 59′ 59 44
                                            360 00 00 00

                                                 00 67
                                                     4
                                                 2″ 01

Corrections		Adjusted Angles		
$(1) = -0'' 39$	$54°$	$38'$	$00'' 23$
$(2) = +0'' 26$	$50°$	$35'$	$04'' 73$
$(3) = +0'' 13$	74	46	$56'' 25$
$(4) = -0'' 24$	78	$13'$	$34'' 25$
$(5) = +0' 18$	51	$53'$	$13'' 53$
$(6) = +0'' 06$	$49°$	$53'$	$13'' 48$
$(7) = -0'' 31$	86	$22'$	$00'' 16$
$(8) = +0'' 40$	30	$53'$	$44'' 33$
$(9) = -0'' 09$	60	$44'$	$18'' 98$
$(10) = -0'' 75$	26	$49'$	$17'' 40$
$(11) = +0'' 45$	49	$01'$	$47'' 09$
$(12) = +0'' 30$	104	$08'$	$60'' 00$
$(13) = -0'' 47$	38	$18'$	$05'' 86$
$(14) = +0'' 20$	71	$15'$	$24'' 56$
$(15) = +0'' 27$	70	$26'$	$31'' 29$

APPROXIMATE METHOD OF FINDING THE PRECISION

145   An adjustment may be carried out rigorously so far as finding the values of the unknowns is concerned, but only an approximate value of the m s e of the angles or sides may be thought necessary

In good work the following method will give results nearly the same as those found by the rigorous process

The average value μ' of the m s e of an angle in a triangulation net after adjustment is easily seen from Art 102 to be

$$\mu' = \sqrt{\frac{n - n_c}{n}} \mu$$

where

$n =$ number of angles observed

$n_c =$ number of local and general conditions

$\mu =$ m s e of a measured angle of weight unity

The value of μ is, by the usual formula,

$$\mu = \sqrt{\frac{[pvv]}{n_c}}$$

To find the m s e of a side of a triangle a single chain of the best-shaped triangles between the base and the side is selected, all tie lines being rejected   Then, assuming the base to be exact and the m s e of each adjusted angle to be μ', we have from Ex 9, p 234,

$$\mu^2{}_{\log a_n} = \tfrac{2}{3}\,\mu'^2\,[\,\partial_A{}^2 + \partial_b{}^2 + \partial_A\,\partial_b\,]$$

where ∂_A, ∂_B are the log differences corresponding to $1''$ for the angles A, B in a table of log sines

A form still more approximate was used on the U S Lake Survey in the determination of the precision of a side of the primary triangulation   The angles of each triangle were taken to be independent of one another   In this case evidently (Ex 5, p 109)

$$\mu^2{}_{\log a_n} = \mu'^2\,[\,\partial_A{}^2 + \partial_b{}^2\,]$$

The earlier work of the Coast Survey was computed from this same formula

Ex To find the m s e of the side *OT* as derived from the base *AS* in the figure *ONSL* (Fig 19)

Number of angles measured $= 9$
Number of conditions, local and general, $= 5$
From the adjustment (Art 140) $[\rho v v] = 7\,54$

$$\mu = \sqrt{\frac{7\,54}{5}}$$

$$= 1''\,23$$

$$\mu' = \sqrt{\frac{9 - 5}{9}}\,\mu$$

$$= \tfrac{2}{3}\,\mu$$

The chain of triangles is ONS, OLS

Station	Angles	δ	Sqs	Prods
N Base,	122° 11′ 15″	− 13 2	174 2	
S Base	23° 06′ 05″			− 401 3
Oneota,	34 40′ 40″	+ 30 4	924 2	
S Base,	70 39′ 25″	+ 7 4	54 8	
Oncota,	78° 27′ 05″			260 5
Lester,	30 53′ 30″	+ 35 2	1239 0	
				2251 4

$$\mu_{\log OL} = \sqrt{2251\ 4 \times \tfrac{1}{3}}$$

$= 31\ 6$ in units of the seventh decimal place

Also $OL = 16556$ metres

Hence μ_{OL} is known

The Method of Directions

146 This method is due to Bessel   Various modifications of Bessel's plan of making the observations are used on different surveys   The following is that used on the U S Coast Survey

"In any set, after the objects have been observed in the order of graduation they are re-observed with instrument reversed in the opposite order, the mean of the two observations upon each object is then taken   The number of such sets and the number of positions made depends on the accuracy required and upon the perfection of the instrument"   A single series of means is called an "arc"

"The direction instrument requires that it should be turned on its stand or changed in position, in order that the direction of any one line, and consequently of all, should fall upon different parts of the circle as the only security against errors of graduation   The number of positions

varies from five to twenty-one of nearly equal arcs, and in each position the circuit of the horizon is made, giving the direction of each line by two observations, one in the direct and the other in the reversed position of the telescope These circuits or series are repeated in each position until two to five values of each direction are obtained   Each angle is therefore determined by from 35 to 63 measurements in the direct and a like number in the reversed position of the telescope "

147 The Local Adjustment.—Let O be the station occupied and 1, 2, 3,   the stations sighted at in order of azimuth   Let some one direction, as $O1$, be selected as the zero direction, and let A, B,   denote the most probable values of the *angles* which the directions of the different signals make with this direction

Fig 41
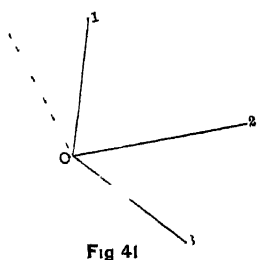
In the first arc let X_1 denote the most probable value of the angle between the zero of the limb of the instrument and the direction of the signal taken as the zero direction, then if M_1', M_1'', denote the readings of the limb for the different signals, and v_1', v_1'',   the most probable corrections to these readings, we have the observation equations

$$X_1 \qquad - M_1' = v_1'$$
$$X_1 + A - M_1'' = v_1''$$
$$X_1 + B - M_1''' = v_1'''$$
.

The zero of the limb being changed in the next arc, we have in like manner

$$X_2 \qquad - M_2 = v_2'$$
$$X_2 + A - M_2 = v_2''$$
$$X_2 + B - M_2 = v_2'''$$
.

and so on for the remaining arcs

If now p_1', p_1'',   , p_2', p_2'',   ,   denote the weights of the measured directions of the several series of arcs, the normal equations follow at once   They are

$$[p_1]X_1 \qquad\qquad +p_1''A+p_1'''B+ \qquad =[p_1M_1]$$
$$[p_2]X \qquad\qquad +p_2''A+p_2'''B+ \qquad =[p\ M]$$

$$p_1''\ X_1+p_2''\ X_2+ \qquad +[p'']A \qquad\qquad =[p\ M\ ]$$
$$p_1'''\ X_1+p_2'''\ X_2+ \qquad\qquad +[p''']B \qquad =[p\ M\ ]$$

from which the unknowns may be found

In order to shorten the numerical work a course similar to that of Art 41 may be followed   Let

$$X_1 = M_1' + r_1 \qquad\qquad A = A' + (A)$$
$$X_2 = M_2' + x_2 \qquad\qquad B = B' + (B)$$

where M_1', M_2', A', B',   are approximate values of X_1, X_2 A, B,   , and x_1, x_2, (A), (B),   denote their most probable corrections

Also, for convenience in writing, put

$$m_1'' = M_1'' - M_1' - A' \qquad\qquad m_2'' = M_2'' - M_2' - A$$
$$m_1''' = M_1''' - M_1' - B' \qquad\qquad m_2''' = M_2''' - M_2' - B$$

The normal equations now become

$$[p_1]x_1 \qquad\qquad +p_1''(A)+p_1'''(B)+ \qquad =[p_1m_1]$$
$$[p_2]x_2 \qquad\qquad +p_2''(A)+p_2'''(B)+ \qquad =[p_2m]$$

$$p_1''\ x_1+p_2''\ x_2+ \qquad +[p''](A) \qquad\qquad =[p'\ m']$$
$$p_1'''x_1+p_2'''x_2+ \qquad\qquad +[p'''](B) \qquad =[p'\ m\ ]$$

The quantities z_1, z_2, ... being merely auxiliary quantities, we eliminate them by substituting their values as found from the first group of normal equations, in the second group. We have then

$$\ldots$$

$$\ldots$$

which may be solved as usual.

These equations may be written

$$[aa][(A)] + [ab][(B)] + \ldots = [\,]$$
$$[ab][(A)] + [bb][(B)] + \ldots = [\,]$$
$$\ldots$$

where $[aa]$, $[ab]$, ... are to be looked on as mere symbols.

In the cases that occur in ordinary work the computation may be still farther shortened. If we arrange the observations in groups containing readings on the same series of signals, then, these readings being of equal value, we have for the first group

$$p_1' \quad p_2' \quad \ldots \quad p \text{ suppose}$$
$$p_1' \quad p \qquad \quad \rho \quad ''$$

and therefore

$$[p] = n'.p_1 \quad [p] = n'.p_1$$

n' being the number of signals sighted at in this group. Similarly for the other groups.

Hence if n_a', n_a'',   denote the number of arcs in the several groups, the coefficients of the normal equations become

$$[aa] = \lfloor p'' \rfloor - \frac{n_a'}{n_s'} {}_1 p'' - \frac{n_a''}{n_s''} {}_1 p'' -$$

$$[ab] = - \frac{n_a'}{n_s'} {}_1 p'' {}_1 p''' - \frac{n_a''}{n_s''} {}_1 p'' {}_1 p''' -$$

$$[al] = \lfloor p'' m'' \rfloor - \frac{\lfloor m_1 \rfloor}{n_s'} {}_1 p'' - \frac{\lfloor m_1 \rfloor}{n_s''} {}_1 p'' -$$

where ${}_1 p'$, ${}_1 p''$,   denote any of the equal weights in the several columns of the first group, and ${}_1 p'$, ${}_1 p''$,   , ${}_1 p'$, ${}_1 p''$,   , denote corresponding quantities in the second, third, groups These weights for the signals sighted at may be taken to be each equal to unity, and, for the signals not sighted at, zero

After having found the quantities m', m'', by taking the differences between the approximate values of the angles and the several measured values, it is convenient to arrange the formation of the normal equations according to the following scheme

No of Group	n_a n_s	p	$p'' m''$	$p''' m''$		sum	sum n_s
1							
2							
3							
		$\lfloor p \rfloor$	$\lfloor p'' m'' \rfloor$	$\lfloor p''' m''' \rfloor$		$\lfloor pm \rfloor$	

The coefficients of the normal equations may now be written down at sight

Special Case. If every arc is full, and every direction is equally well measured in every arc, then

$$| p | | p | n,$$
$$| p_1 | | p | n$$

and the normal equations become

$$\left(n, \ \frac{n_a}{n} \right)(A) \quad \frac{n_a}{n_1}(B) \qquad \qquad (m) \quad \frac{n}{n}$$

$$\frac{n_a}{n_1}(A) + \left(n, \ \frac{n_1}{n} \right)(B) \qquad \qquad (m) \quad \frac{n}{n}$$

$$. \qquad \qquad .$$

By addition, the number of arcs being n 1,

$$\frac{n_a}{n_1}\left((A) + (B) + \right) \qquad \right) \quad \frac{| m |}{n}$$

Hence

$$n_a(A) \quad | m |$$
$$n_a(B) \quad | m |$$
$$. \qquad .$$

as is evident *a priori*

148. *Checks of the Normal Equations*
1. The sum

$$[aa] + [ab] + [a] +$$
$$+ [bb] + [b] +$$
$$+ [c] +$$
$$+$$

is equal to half the number of observations, less half the number of arcs.

For substituting for $[aa]$, $[ab]$,   their values we find the above sum equal to

$$[p''] + [p'''] +$$
$$- \frac{p_1''^2 + p_1''' + p_1''p_1''' +}{[p_1]} \qquad - \frac{p_2'' + p'''^0 + p_2''p_2'' +}{[p_2]} -$$

$$- \tfrac{1}{2}\{[p'] + [p''] + [p''] + \qquad \{$$
$$\quad - p_1' + p_1'' + p_1''' + \qquad - \frac{p_1'' + p_1'''^0 + p_1''p_1''' +}{[p_1]} -$$
$$\qquad\qquad 2$$

$$\quad - p_2'' + p_2'' + p_2''' + \qquad - \frac{p''^2 + p_2'''^0 + p_2'p_0''' +}{[p_2]} -$$
$$\qquad\qquad 2$$
$$+$$

$$- \tfrac{1}{2}\{[p'] + [p''] + [p'''] + \qquad \} - \frac{[p_1^0]}{2[p_1]} - \frac{[p_2']}{2[p_2]} -$$

Now, $p_1' = p_2' = \quad = 1$, and therefore

$$[p'] + [p''] + [p'''] + \qquad = \text{the number of observations}$$

$$\frac{[p_1]}{[p_1]} + \frac{[p_2']}{[p_2]} + \frac{[p_3']}{[p_1]} + \qquad = \text{the number of arcs}$$

2 The sum

$$[al] + [bl] + [cl] + \qquad = [wl]$$

where $[wl]$ is formed in the same way as $[al]$, $[bl]$,

For

$$[al] + [bl] +$$
$$= |p''m''| + |p'''m'''| +$$
$$+ |p_1 m_1|\left(\frac{p_1'}{|p_1|} - 1\right) + |p_2 m_2|\left(\frac{p_2'}{|p_2|} - 1\right) +$$
$$= \frac{p_1'}{|p_1|}|p_1 m_1| + \frac{p_2'}{|p_2|}|p_2 m_2| +$$

since $|p''m''| + |p'''m'''| +$
$$- |p_1 m_1| - |p_2 m_2| - \qquad = [p'm']$$
$$= 0$$

which proves the proposition

149. *The Precision of the Azimuth Lines.* — The last step is to find p, the basis of an azimuth direction.

We have generally

$$p = \frac{\text{No. of obs. quant.} - \text{No. of independent unknown}}{}$$

Now, from the observation equations, Art. 143, it is evident that the number of independent unknowns is equal to the number of arcs n_a, together with the number of stations sighted at, less one. Hence

$$p = n + (n_1 + n - 1)$$

n being the number of directions.

To compute the value of $[p v v]$ I am to derive z, z_1 from the normal equations, Art. 143, and we have the reduced normal equations

$$[p b] = [p c] + [p d] + [p e] \quad , \quad \cdots$$
$$[p b] = [p c] + [p d] + [p e] \quad \cdots \quad , \quad \cdots$$

$$[a a][c] + [a v][d v] \quad [v]$$
$$[b b][c d] \quad [v v]$$

Hence, as in Art. 106,

$$[p v v]$$

$$[p u u] \quad \frac{[p m_1]^2}{[p]} \quad \frac{[p m_1]^2}{[p]} \quad [a d]^2 \quad [d d]$$
$$\frac{}{[p]} \quad \frac{}{[p]} \quad [a a]^2 \quad [c c]$$

$$[p u u] \quad \frac{[p m_1]^2}{[p]} \quad \frac{[p m_1]^2}{[p]} \quad \cdots \quad [a][c d] \quad [d][p]$$
$$\frac{}{[p]} \quad \frac{}{[p]}$$

The quantity $[p u u]$ should always, if possible, be found from the original observations. It will in general be quite different if found from the means forming the different groups of arcs taken as single observations. (See Ex., Art. 62.)

Taking the weight of each observed direction to be o the same value unity, the expressions for $[vv]$ may be written

$$[vv] = [mm] - \frac{[m_1]^2}{n_s'} - \frac{[m_o]^2}{n_s''} - \quad - \frac{[al]^2}{[aa]} - \frac{[bl\ 1]^2}{[bb\ 1]} -$$

$$= [mm] - \frac{[m_1]^2}{n_s'} - \frac{[m_.]^2}{n_s''} - \quad - [al](A) - [bl](B) -$$

150 **The General Adjustment.**—The general adjustment is carried out as in the case of independent angles The angle and side equations are formed as in Arts 117–139, and the solution effected according to the programme of Art 113

151 *Ex* At station Clark Mt, in the triangulation of the Blue Ridge, Va, readings were made with a non-repeating theodolite in the method of arcs The following, taken from these readings, will be sufficient to illustrate the method of reduction

The original observations are arranged in sets containing readings on the same groups of signals, and the quantities given in the table below are the remainders found by subtracting the reading of the first direction in each series from the readings of the other directions, that is, $M_1'' - M_1'$, $M_1''' - M_1'$,

At Clark Mt

Spear	Humpback	Fork	
00 00′ 00″ 00	24° 09′ 35″ 70	78° 26′ 08″ 55	
00	39″ 40	09″ 60	
00	36″ 23	09″ 33	
00	33″ 55	10″ 45	
00″ 00′ 00″ 00		78° 26′ 10″ 20	
00		11″ 03	
00′ 00′ 00″ 00	24° 09′ 36″ 10		
00	37″ 40		
00	36″ 53		
00	38″ 15		
00	39″ 00		
	00″ 00′ 00″ 00	51° 16′ 31″ 85	
	00	31″ 94	
	00	32″ 03	
	00	36″ 19	
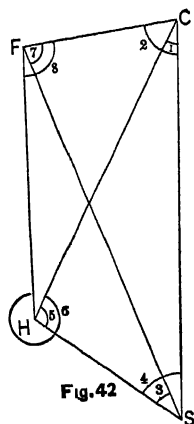
Fig.42

The Local Adjustment

Assume the most probable values of the angles

Spear,	00" 00' 00" 00
Humpback,	24" 00' 36" 00 + (A)
Fork,	78' 26' 9" 00 + (B)

Take the differences between the approximate values of the angles and the several measured values We then have

$p'm'$	$p''m''$	$p'''m'''$	Sums
$n_a' = 4$ 0 00	$_1p'' = 1$ $-$ 1 20	$_1p''' = -1$ $-$ 1 35	
00	$+$ 2 50	0 30	
$n_s' = 3$ 00	$-$ 0 67	0 57	
	$-$ 3 35	$+$ 0 55	
0 00	$-$ 2 72	1 07	$+$ 10
$n_a'' = 2$ 0 00			
$n_s'' = 2$ 00	$_2p'' = 0$	$_2p'''$ 1 $+$ 0 30	
		$+$ 1 13	
0 00		$+$ 1.43	$+$ 1 43
0 00	$-$ 0 80		
00	$+$ 0 50		
00	$-$ 0 37		
00	$+$ 1 25		
00	$+$ 2 10		
0 00	$+$ 2 68		$+$ 2 68
	0 00	1 15	
	00	$-$ 1.00	
	00	$-$ 0 97	
	00	$+$ 3 13	
	0 00	$-$ 0 05	0 05
$[p'm'] = 0$ 00	$[p''m''] = -$ 0 04	$[p'''m'''] = -$ 0 29	$[pm]$ 0 33
$[p'] = 11$	$[p''] = 13$	$[p'''] = 10$	

Next form the table

No of group	$\dfrac{n_a}{n_s}$	n_a	$p'm'$	$p''m''$	$p'''m'''$	Sum	$\dfrac{\text{Sum}}{n_s}$
1	$\frac{1}{1}$	4	0	− 2 72	− 1 67	− 4 39	− 1 463
2	$\frac{3}{2}$	2	0		+ 1 43	+ 1 43	+ 0 715
3	$\frac{5}{2}$	5	0	+ 2 68		+ 2 68	+ 1 340
4	$\frac{1}{2}$	4	0	00	− 0 05	− 0 05	− 0 025
Sums,		15		− 0 04	− 0 29	− 0 33	Check

The coefficients of the normal equations

$$\left[\,aa\,\right] = 13 - \tfrac{1}{1} - \tfrac{5}{2} - \tfrac{1}{2} = + 7\tfrac{1}{6}$$

$$\left[\,ab\,\right] = \qquad - \tfrac{4}{8} - \tfrac{1}{2} \qquad = - 3\tfrac{1}{8}$$

$$\left[\,bb\,\right] = 10 - \tfrac{4}{8} - \tfrac{9}{2} - \tfrac{1}{2} = + 5\tfrac{2}{4}$$

$$\left[\,al\,\right] = - 0\ 04 + 1\ 463 - 1\ 34 + 0\ 025 = + 0\ 108$$

$$\left[\,bl\,\right] = - 0\ 29 + 1\ 463 - 0\ 715 + 0\ 025 = + 0\ 483$$

$$\left[\,wl\,\right] = - 1\ 463 + 0\ 715 + 1\ 34 \qquad = + 0\ 592$$

Check (1) $\left[\,aa\,\right] + \left[\,ab\,\right] + \left[\,bb\,\right] = 9\tfrac{1}{2}$

$$= \tfrac{1}{2}(34 - 15)$$
$$= \tfrac{1}{2}(n - n_a)$$

as it should

(2) $\qquad\qquad [al] + [bl] = 0\ 591$
$$= [wl]$$

as it should

The normal equations are

$$7\tfrac{1}{6}(A) - 3\tfrac{1}{8}(B) = + 0\ 108 = [al]$$
$$- 3\tfrac{1}{8}(A) + 5\tfrac{3}{4}(B) = + 0\ 483 = [bl]$$

The general solution of these equations gives

$$(A) = + 0\ 1921\,[al] + 0\ 1130\,[bl]$$
$$(B) = + 0\ 1130\,[al] + 0\ 2429\,[bl]$$

Substituting for $[al]$, $[bl]$ their values, there result

$$(A) = + 0''\ 075$$
$$(B) = + 0''\ 130$$

and hence the local directions

$$
\begin{array}{llll}
\text{Spear} & 0° & 00 & 00''\,000 \\
\text{Humpback,} & 24° & 09' & 36''\,975 \\
\text{Fork,} & 78° & 26' & 10''\,030
\end{array}
$$

To find the m s e of a single observation

The value of $[vv]$, computed according to Art 150, is found to be 25 7
Therefore

$$
\begin{aligned}
\mu &= \sqrt{\dfrac{25\ 7}{34 - (15 + 3 + 1)}} \\
&= 1''\,2
\end{aligned}
$$

the divisor in this case being 15

This completes the local adjustment at this station   Proceed similarly at the remaining three stations

The General Adjustment

At Clark

Most probable directions

$$
\begin{array}{llll}
\text{Spear,} & 0° & 00' & 00''\,000 \\
\text{Humpback,} & 24° & 09' & 36''\,975 + (1) \\
\text{Fork,} & 78° & 26' & 10''\,030 + (2)
\end{array}
$$

Weight Equations

$$(1) = + 0\ 1921\ \boxed{1} + 0\ 1130\ \boxed{2}$$
$$(2) = + 0\ 1130\ \boxed{1} + 0\ 2429\ \boxed{2}$$

$$[vv] = 25\ 7 \qquad \text{Divisor, 17}$$

At Spear

$$
\begin{array}{llll}
\text{Humpback,} & 0° & 00' & 00''\,000 \\
\text{Fork,} & 32° & 08' & 11''\,793 + (3) \\
\text{Clark,} & 54° & 06' & 29''\,197 + (4)
\end{array}
$$

Weight Equations

$$(3) = + 0\ 2061\ \boxed{3} + 0\ 0485\ \boxed{4}$$
$$(4) = + 0\ 0485\ \boxed{3} + 0\ 1879\ \boxed{4}$$

$$[vv] = 58\ 7 \qquad \text{Divisor, 17}$$

At Humpback

Clark,
Spear,
Fork,

Weight Equations

Divisor,

At Fork.

Clark,
Spear,
Humpback,

Weight Equations

Divisor,

The angle and side equations are formed as already explained. The angle equation from the triangle SFC and the side equation from the quadrilateral CSFH (pole at C) will be found to be

From this point the solution is carried through exactly as in Art. 190. The finally adjusted directions will be found to be

At Clark, Spear,   At Humpback, Clark,
        Humpback,   Spear,
        Fork,   Fork,

At Spear, Humpback,   At Fork, Clark,
        Fork,   Spear,
        Clark,   Humpback,

The m. s. e. of an observation of weight unity is
The m. s. e. of the adjusted value of the angle CSF

The form of reduction used on the U. S. Coast Survey for the adjustment of the primary triangulation is essentially the same as that just explained, so far as finding the values of the local corrections is concerned   The method of weighting employed in the general adjustment is not given, as it is not so satisfactory as that in the text.   It will be found in *Report*, 1864, app 14

152 **Approximate Method of Reduction.**   A very convenient method of approximation may be derived from the normal equations, Art. 147.   It depends on the theorem of Art 115, and hence, if the process is repeated often enough, leads to the same result as the rigorous solution

In the second group of normal equations, Art. 147, assume

$$X_1 - M_1', \quad Y_1 \quad M_2', \ldots$$

then we have as approximate values of $A, B, $ . .

$$A' = \frac{[p''(M'' - M')]}{[p'']}, \quad B' \quad \frac{[p'''(M''' \quad M')]}{[p''']} \quad \ldots$$

Let r_1 denote the correction to Y_1, r_2 to Y_2 .   Then the values $A', B', $ of $A, B, $. substituted in the first group of normal equations, give as approximate values of $x_1, x_2,$.

$$x_1' = \frac{p_1''(M_1'' - M_1'   A') + p_1'''(M_1''' \quad M_1' \quad B) + \quad \ldots}{[p_1]}$$

$$x_2' = \frac{p_2''(M_2'' \quad M_2' \quad A') + p_2'''(M_2''' \quad M' \quad B) + .}{[p_2]}$$

·   ·   ·   ·   ·   ·   ·   ·   ·   ·   ·

which values substituted in the second group give as second approximations to the values of $A, B, $.

$$A'' = \frac{p_1''(M_1'' - M_1' \quad x_1') + p_2''(M_2'' \quad M_2' \quad r_1 + \ldots}{[p'']}$$

$$B'' = \frac{p_1'''(M_1''' - M_1' - x_1') + p_2'''(M_2''' \quad M' \quad r_1 + \ldots}{[p''']}$$

·   ·   ·   ·

and so on.

This approximate form of reduction was used on the Ordnance Trigonometrical Survey of Great Britain in the reduction of the principal triangulation   The approximations were carried out only as far as A'', B'',   This is in general sufficient in good work

Instead of finding A', B',   as above, it is often more convenient to follow the Ordnance Survey plan, and, instead of deducting M_1', M_2'',   from the readings of the signals in the different arcs in order, to add to the readings of the signals in the different arcs quantities which will make the column M' constant throughout   We should then have

$$A' + M' = \frac{\lfloor p''M''\rfloor}{\lfloor p''\rfloor}, \quad B' + M' = \frac{\lfloor p'''M'''\rfloor}{\lfloor p'''\rfloor}, \ \cdot$$

The corrections τ and the other approximations are made as before

Should the observation of the zero direction be wanting in any arc—as, say, the third—the quantity to be added to each reading in this arc is given approximately by the value of X_3 in the first set of normal equations   Thus, given the readings prepared as explained above,

$$M', \ M_1'', \ M_1'''$$
$$M', \ M_2'', \ M_2'''$$
$$\text{o,} \quad M_3''' - M_1''$$

to find $M_1'' - M_1'$

If A', B' have been found from the first two arcs, then from the normal equations

$$M_3' = X_3 = \frac{p_3'M_1' + p_3''(M_3'' - A') + p_3'''(M_3''' - B')}{\lfloor p_3 \rfloor}$$

and

$$M_1'' - M_1' = \frac{p_1''A' - p_1'''(M_1''' - M_3'' - B')}{p_1'' + p_1'''}$$

$$= A' \text{ approx}$$

330 — THE ADJUSTMENT OF OBSERVATIONS.

Hence the complete form would be

Modified Rigorous Solution

154   The forms that have been given in the preceding articles for the rigorous adjustment of a triangulation, though analytically very elegant, are somewhat complicated.   A method which shall give a marked diminution of work in the reduction without increasing the field work materially is a desideratum.

In the reduction of a long net of triangulation we have seen that labor is eased by breaking the work into two parts, first adjusting at each of the stations for the local conditions, and then using this work in the further adjustment arising from the angle and side equations.   Now, if the measurements were made on a uniform plan the local adjustment would be simplified, as we should have a similar problem to solve at each station.   This would lead to a great saving of labor, since, if the measurements are made at haphazard, the local adjustment may be quite complicated.

If we decide, then, that the observer must work in accordance with some regular form, our next inquiry is, What shall that form be?   First, shall the angles be measured independently or in arcs?   The point to be aimed at in this as in all work of precision is to get rid of systematic error.   The accidental errors are trifling in comparison.   When we consider twist of triangulation station from the action of the sun's rays, the influence on distinctness of vision for the same focus for different lengths of lines sighted over, the interruptions that may occur in the course of reading a long arc; the more uniform light that may always be had when the number of signals in use at one time is small, etc., we cannot but conclude that greater precision is to be attained by measuring the angles independently.   The errors are more likely to mutually balance.   Even Andræ, the author of the most important contributions to the method of directions since Bessel, and who used this method in the triangulation of Denmark, acknowledges

that "in place of observations of directions in arcs it is preferable to return to the old method of Gauss in measuring angles "+

As regards the cost, it must be acknowledged that for an equal number of results, leaving quality out of account, the method of arcs has the advantage   Nowadays, however, when facilities exist for measuring angles by night as well as by day, there is less delay in waiting for suitable conditions than when day work alone had to be depended on †   Taking this into account, the difference in cost would not be great in any case, more especially as a triangulation party is never a very large one

Having decided that angles should be measured independently, it is in accordance with general experience that instead of spending all of the time of observation in measuring the single angles themselves better results would be obtained by spending part of it in measuring combinations of the angles   A simple form that at once suggests itself would be to close the horizon at each station, that is, to measure all of the angles $AOB, BOC,$ LOA in order round the horizon (see Fig 20)   The local corrections would each be $\frac{1}{n}$ of the discrepancy of the sum of the angles from 360° (Art 121), and the reduction is thus simple and uniform   However, as measuring the closing angle LOA is the same as measuring the sum angle AOL, it would seem that if we measure one sum angle we ought to measure all possible sum angles ‡   Though the form of adjustment for this combination of measures is a special case of that already given, I shall, at the risk of a little repetition, sketch it in full

154 The Local Adjustment (Angles). Let O be the station occupied, and $1, 2, 3, 4$ the stations related it in order of azimuth; then the angles to be measured would be

$$1o2$$
$$1o3 \qquad 2o3$$
$$1o4 \qquad 3o4 \qquad 3o4$$

Fig 43
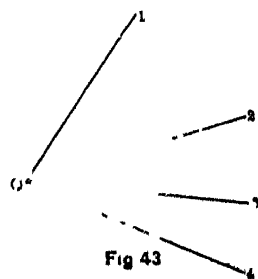
Take the first three as independent unknowns, and let A, B, C denote their most probable values. Also let $l_1, l_2 \ldots$ denote the several measured values.

The observation equations are

$$
\begin{array}{llll}
A & & l_1 & c_1 \\
& B & l_2 & c_2 \\
& \quad C & l_3 & c_3 \\
A + B & & l_4 & c_4 \\
A \quad + C & l_5 & c_5 \\
B + C & l_6 & c_6
\end{array} \tag{1}
$$

and the normal equations

$$
\begin{array}{llllll}
3A & B & C & l_1 & l_4 & l_5 \\
A + 3B & C & l_2 + l_4 & l_6 \\
A & B + 3C & l_3 + l_5 + l_6
\end{array} \tag{2}
$$

Solve the c equations, and

$$
\begin{array}{l}
5A \quad (l_1 + l_4 \quad l_2) + (l_5 \quad l_6) \\
5B \quad (l_2 + l_4) + 2l_5 + (l_6 \quad l_1) \tag{3} \\
5C \quad (l_3 + l_5 + (l_6 \quad l_4) + 2l_6
\end{array}
$$

It is useful to notice as a check that

$$A + B + C \quad l_1 + l_2 + l_3 \tag{4}$$

For practical computation arrange in tabular form as follows. Sum first the horizontal rows, next form the sum

differences, and then place these quantities in the proper columns   Add each column and divide the sums by 4

l_{12}	l_{13}	l_{14}	Sum_1	
	l_{23}	l_{24}	Sum_2	$\mathrm{Sum}_1 - \mathrm{Sum}_2$
		l_{34}	Sum_3	$\mathrm{Sum}_1 - \mathrm{Sum}_3$
$\mathrm{Sum}_1 - \mathrm{Sum}_2$	$\mathrm{Sum}_1 - \mathrm{Sum}_3$	Sum_1		
$4A$	$4B$	$4C$		
A	B	C	Check	

Check Solution —The form of the expressions found as the values of A, B, C suggests a method of finding these values in accordance with the fundamental principle of the mean   For, writing the value of A in the form

$$A = \frac{2l_{12} + (l_{13} - l_{23}) + (l_{14} - l_{24})}{2 + 1 + 1} \qquad (5)$$

we see that it is the weighted mean of the values

$$
\begin{array}{ccc}
l_{12} & \text{weight} & 1 \\
l_{13} - l_{23} & \text{``} & \tfrac{1}{2} \\
l_{14} - l_{24} & \text{``} & \tfrac{1}{2}
\end{array}
$$

that is, the value of A is found by taking the weighted mean of the measured value of A and of all the values of A that can be found by combining two other measures
Similarly for the remaining angles
The Precision —From Eq 3 or from Eq 5 it is evident that each of the angles A, B, C has the weight 2   The weight P of the angle $- A + B$ is given by (see Art 101)

$$\frac{1}{P} = [\alpha\alpha] - 2[\alpha\beta] + [\beta\beta]$$
$$= \tfrac{2}{4} - 2\tfrac{1}{4} + \tfrac{2}{4}$$
$$= \tfrac{2}{4}$$
$$\text{and} \quad P = 2$$

The same result will be found for the remaining angles Hence after the station adjustment each angle has the same weight, being double the weight of a single measured angle in our case of 4 stations sighted at   With n stations sighted at the weight of an adjusted angle would be $\frac{n}{2}$ times the weight of a single measured angle

Ex 1 Given at station Oswego, in the triangulation of Lake Ontario (Fig 35),

l_{12}	80	29′	46″ 10
l_{13}	107	19′	03″ 28
l_{14}	138	12	49″ 44
l_{23}	26	49′	16′ 61
l_{24}	57	43′	01′ 96
l_{34}	30′	53′	42′ 88

required the adjusted values

Solution

46 10	63 28	109′ 44	218″ 82	
	16 61	61′ 96	78″ 57	140″ 25
49 44	1, 44	42″ 88	42″ 88	175″ 94
49 44	255 53	218′ 82		
16 60	63 96	133′ 10	218″ 83	check
		105″ 28		
	B	C		

and the adjusted angles

80	29′	46″ 59
107	19′	03″ 96
138	12′	48″ 28
26	49′	17″ 37
57	43′	01″ 60
30	53′	44″ 32

Check Solution

Angle A		Angle B		Angle C	
46 10 wt 1		1′ 28 wt. 1		49″ 44 wt 1	
46 67 " ½		2′ 71 " ½		46″ 16 " ½	
17′.45 " ½		6″ 56 " ½		48″.06 " ½	
Mean, 46 59		3′ 96		48″ 28	

Similarly for the remaining angles.

   The weight of each adjusted angle is twice that of each measured angle

Ex 2. The angles at station Vanderlip (Art. 142) may be adjusted in the same way as the above

155 The General Adjustment (Angles).—The angle and side equations are formed as explained in Arts 117–139 The connection between the local and general adjustment is through the weight equations In our example of 4 stations, in which every angle between every two directions is measured, the weight equations are in three groups, of which the first is (see Eq 2, p 333)

$$3[\sigma a] - [a\beta] - [a\gamma] = 1$$
$$- [a\sigma] + 3[\sigma\beta] - [\sigma\gamma] = 0$$
$$- [aa] - [a\beta] + 3[\sigma\gamma] = 0$$

which equations are moderately simple in form

If we had simply closed the horizon the first group of weight equations would have been

$$2[a\sigma] + [a\beta] + [a\gamma] = 1$$
$$[a\sigma] + 2[\sigma\beta] + [\sigma\gamma] = 0$$
$$[\sigma a] + [\sigma\beta] + 2[\nu\gamma] = 0$$

which are as complicated as the preceding

Hence, in a net in which the angles at each station are measured in either of the ways indicated, the solution would be simplified so far as the local adjustment is concerned, in that the adjustment at each station is of a fixed form, but in the general adjustment arising from the angle and side equations the gain is comparatively little

If, however, instead of finding the corrections to the angles we had found the corrections to the directions of the arms of the angles, the weight equations become much simplified, and therefore also the whole reduction This idea, which, I believe, is due to Hansen, will now be developed *

156 The Local Adjustment (Directions).—If X, A, B, C denote the readings of the 4 directions 1, 2, 3, 4 from O, then since $A - X$, $B - X$, $C - X$ correspond to the

---

* See *Die preussische Landestriangulation* Berlin, 1874, seq

A, B, C in the angle adjustment, the observation equations may be written

$$
\begin{aligned}
X &- A && && & l_1 & c_1 \\
X && + B && && l_2 & c_2 \\
X &&& + C && & l_3 & c_3 \\
& A & + B && && l & c_4 \qquad (1) \\
& A &&& + C && l_5 & c_5 \\
&& B & + C && l_6 & c_6
\end{aligned}
$$

and the normal equations

$$
\begin{aligned}
3X & - A & - B & - C && l_1 & l_2 & l_3 \\
X & + 3A & - B & - C && l_1 & l & l_5 \\
X & - A & + 3B & - C && l_2 + l_4 & l_6 \\
X & - A & - B & + 3C && l_3 + l_5 + l_6 && (2)
\end{aligned}
$$

Adding these equations, there results

$$ 0 \qquad 0 $$

and therefore the unknowns cannot be found without some further relation connecting them. The reason of the indeterminate form is that directions are nothing but the angles which the rays $c_1, c_2, \ldots$ make with some common zero ray whose position is not fixed, and which may therefore be taken arbitrarily. To carry out the solution it will be most convenient to fix the zero ray by making the arbitrary assumption

$$ X + A + B + C = 0 \qquad (3) $$

By adding this to each of the normal equations they reduce to

$$
\begin{aligned}
4X && l_1 & l_2 & l_3 \\
4A && l_1 & l_4 & l_5 \\
4B && l_2 + l_4 & l_6 & \qquad (4) \\
4C && l_3 + l_5 + l_6
\end{aligned}
$$

which give the values of X, A, B, C directly.

The computation may be rendered quite mechanical by arranging in tabular form

1	2	3	4	Sums
	l_{12}	l_{13}	l_{14}	Sum_1
		l_{23}	l_{24}	Sum_2
			l_{34}	Sum_3
$-Sum_1$	$-Sum_2$	$-Sum_3$		
$4X$	$4A$	$4B$	$4C$	
X	A	B	C	Check

The transformation of the normal equations (2) into (4) by means of the arbitrary relation (3) is allowable   For since the sum of the coefficients of the unknowns in each of equations 2 is zero, whatever values of X, A, B, C satisfy those equations $X+a$, $A+a$, $B+a$, $C+a$, where a is any constant, will also satisfy them   Hence whatever set of values is taken to satisfy the equations, the differences $A-X$, $B-X$, $C-X$ will be the same in value   Therefore by arbitrarily fixing the zero direction we find determinate values for the corrections to the other directions

Ex   Take that of Ex 1, Art 154   The tabular scheme, writing down the seconds only, is

	$46''$ 10	$63''$ 28 $16''$ 61	$109''$ 44 $61''$ 96 $42''$ 88	$218''$ 82 $78''$ 57 $42''$ 88
$-218''$ 82	$-78''$ 57	$-42''$ 88		
$-218''$ 82 $-54''$ 70 X	$-32''$ 47 $-8''$ 12 A	$37''$ 01 $9''$ 25 B	$214''$ 28 $53''$ 57 C	

and $A-X$, $B-X$, $C-X$ give the same values of the angles as found before Hence, whether we find the local corrections to the angles or to the directions

of the arms of the angles, the result is the same   One method or the other may, therefore, be used, as is most convenient

To avoid the use of large numbers certain approximate values may be assumed for the directions, and the corrections to these approximate values found   (Compare Art 81)   Thus if (X), (A), (B), (C) denote the corrections to assumed values of X, A, B, C, we may proceed as follows

Assumed approximate angles

Sodus,	0° 00'	00″ + (X)
Vanderlip,	80° 29'	46″ + (A)
Duck,	107° 19'	03″ + (B)
Stony Pt ,	138° 12'	49″ + (C)

Tabular Form

1	2	3	4	Sums
− 0″ 82	+ 0″ 10 + 1″ 43	+ 0″ 28 − 0″ 39 + 3″ 12	+ 0″ 44 − 1″ 04 − 3″ 12	+ 0″ 82 − 1″ 43 − 3″ 12
− 0″ 82 − 0″ 20 (X)	+ 1″ 53 + 0″ 38 (A)	+ 3″ 01 + 0″ 75 (B)	− 3″ 76 − 0″ 94 (C)	

and the adjusted angles are as before

157 The General Adjustment (Directions).—The form of the normal equations 4, Art 156, shows that at each station the quantities X, A,   are determined independently of one another   The weight of each direction at a station, as shown by the weight equations, is represented by the number of stations sighted at   If, therefore, (1), (2),   denote the corrections to the values X, A, locally adjusted, which arise from the angle and side equations, we may consider X, A,   as independent quantities, all of equal weight and subject to certain rigorous conditions, and proceed to carry out the solution according to the simple form of Art 110   Hence, as the solution breaks into two simple problems of adjusting quantities as if independently observed, we see the advantage of adjusting the directions instead of the angles

44

Ex   The quadrilateral Buchanan, Brulè, Aminicon, Lester, in the triangulation of Lake Superior

		Direction		Measured Angle				Local Adj Angle	
At Buchanan,	2–1		47°	57′	36″ 25			36″ 21
	3–1		97°	26′	41″ 29			41″ 32
	3–2		49°	29′	05″ 15			5″ 11
At Brulè,	5–4		40°	25′	47″ 49			47″ 36
	6–4		71°	58′	27″ 31			27″ 45
	6–5		31°	32′	40″ 22			40″ 09
At Aminicon,	8–7		37°	47′	05″ 51			5″ 17
	9–7		97°	51′	03″ 11			3″ 45
	9–8		60°	03′	58″ 62			58″ 28
At Lester,	11–10		51°	00′	39″ 77			40″ 00
	12–10		92°	43′	49″ 42			49″ 20
	12–11		41°	43′	08″ 97			9″ 20

Fig 44
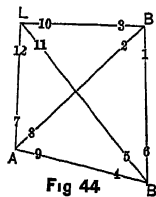
The local adjustment is carried through as in Art 154, it being more convenient to adjust the angles directly rather than the directions

Taking the locally adjusted angles as independent, we next find the corrections to the *directions* arising from the angle and side equations

The angle equations, formed in the usual way from the triangles Buchanan, Brulè, Aminicon ($\varepsilon = 1″$ 37), Brulè, Aminicon, Lester ($\varepsilon = 1″$ 19), Lester, Buchanan, Brulè ($\varepsilon = 1″$ 19), and the side equation, from the quadrilateral itself (pole at Lester), are

$$- (1) + (2) - (4) + (6) - \ (8) + \ (9) = - 0″ \ 57$$
$$- (1) + (3) - (5) + (6) - (10) + (11) = - 0″ \ 22$$
$$- (4) + (5) - (7) + (9) - (11) + (12) = + 1″ \ 18$$

$$- 0 \ 14(1) - 0 \ 90(2) + 1 \ 04(3) + 1 \ 24(4) - 2 \ 95(5)$$
$$+ 1 \ 72(6) + 1 \ 50(7) - 1 \ 36(8) - 0 \ 14(9) = - 0″ \ 80$$

The number of stations pointed at from each station occupied being 3, the weight of each locally adjusted angle is the same throughout the net

Hence we have the correlate normal equations

I	II	III	IV	
+ 6 0	+ 2 0	+ 2 0	+ 0 94	= − 0 57
+ 2 0	+ 6 0	− 2 0	+ 5 84	= − 0 22
+ 2 0	− 2 0	+ 6 0	− 5 84	= + 1 18
+ 0 94	+ 5 84	− 5 84	+ 19 22	= − 0 80

I = − 0 273	III = + 0 377
II = + 0 136	IV = + 0 045

and the corrections

(1)	+ 0′ 11	(7)	− 0″ 31
(2)	0 31	(8)	+ 0″ 21
(3)	+ 0 15	(9)	+ 0″ 10
(4)	0 05	(10)	0″ 14
(5)	+ 0 11	(11)	0 14
(6)	0 06	(12)	+ 0 38

On the Breaking of a Net of Triangulation into Sections for Convenience of Solution.

158. In a long chain of triangulation or in a complicated net the simultaneous solution of the condition equations would be very troublesome, not from any principle involved, but from its very unwieldiness. Accordingly it is necessary to break the work into sections and solve each section by itself. As this breaking into sections causes more or less disturbance of the local conditions at the lines of breaking off, each section should be as large as can be conveniently managed, and the lines of breaking off should be so chosen as to disturb as few conditions as possible. By the method of adjustment explained in Arts. 153–157 much larger sections can be taken than by any other. This is a strong argument in favor of its use.

The contradictions that occur at the lines of breaking off of the several sections are most conveniently bridged over by means of the principle of Art. 115. As an example we may cite the reduction of the principal triangulation of the British Ordnance Survey. There were 920 condition equations to be satisfied in the net. The following was the method of solution employed. [*]

"The triangulation was divided into a number of parts or figures, each affording a not unmanageable number of equations of condition. One of these being corrected or computed independently of all the rest, the corrections so obtained were substituted (so far as they entered) in the

equations of condition of the next figure, and the sum of the squares of the remaining corrections in that figure made a minimum   The corrections thus obtained for the second figure were substituted in the third, and so on "

In the triangulation of Mecklenburg* this method was carried out even more systematically   The adjustment was divided into five groups of 22, 22, 22, 21, 22 condition equations respectively   The corrections resulting from the solution of group I were carried, so far as they entered, into group II , and this group solved, and so on through groups III , IV , V   The whole operation was repeated four times, and the small contradictions still remaining were distributed empirically

For an interesting conference on the whole question see *Comptes Rendus de l'Association Géodésique Internationale*, 1877

Adjustment of a Triangulation for Closure of Circuit

159 In the adjustment of a triangulation we have so far considered it only with reference to a single measured base We have seen how at each station the discrepancy arising from sum angles and from closure of the horizon can be got rid of, and also how in a net joining several stations the conditions arising from closure of triangles and from equality of lengths of sides computed by different routes can be satisfied   There remains the question as to the mode of procedure when several bases enter whose lengths are known and whose positions• have been fixed astronomically   Special cases would be where a circuit of triangulation closed on the initial line, and where a secondary system is to be made to conform to two lines in a primary system, the primary lines being assumed to be known in length and position

The conditions to be satisfied in the adjustment are four in number —that the value of a base computed from another

should agree with the measured value in azimuth, in length, and in latitude and longitude of one of the end points

The measured angles of the triangles connecting the bases having been already adjusted with reference to one base for the local and general conditions, the additional corrections necessary to satisfy the closure of the circuit will be small But little difference in the results will therefore be found by making a simultaneous solution of all of the condition equations of closure and a solution by successive approximation according to the method of Art 115

We shall consider only a single chain of triangles, all tie lines of the system being rejected This on account of simplicity, and also for the reason stated above, that, in good work the corrections being small, it gives results practically close enough—nearly the same, in fact, as a rigorous solution If thought necessary in any special case a rigorous solution of the condition equations can be carried out by the method of correlates

160 Adjustment for Discrepancy in Azimuth.— Let 1-2 and 5-6 be two bases connected through intermediate stations 3, 4 by a single chain of the best-shaped triangles that can be selected from the net
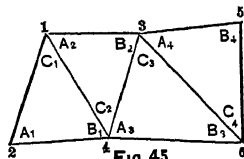
In computing the base 5-6 from 1-2 the sides 1 4, 3-4, 3-6 are at once sides of continuation and bases, according to the triangles considered For example, in the triangle 1 2 4 the side 1-4 is a side of continuation from the base 1-2, but in 1 3 4 the side 1-4 is a base with reference to 3-4 as a side of continuation

In the chain of triangles let

$A_1, A_2,$ be the angles opposite to the sides of continuation

$B_1, B_2,$ the angles opposite to the bases in order of computation

$C_1, C_2,$ the angles opposite to the flank sides

Let z, z_1 denote the values of the measured azimuth of the bases 1–2, 5–6 respectively These values are assumed to be correct, and receive no change in the adjustment

A geodetic computation[x] of the azimuth of the base 5 6 from the base 1–2 is now made, using the values of the angles of the intervening triangles resulting from the adjustment for local and general conditions Call the value of the azimuth of 5–6 computed in this way z'.

Now, reckoning azimuth in the usual way from the south as origin, and the direction of increase from south to west, it is easily seen, by passing from 1 2 along the sides 1–4, 4–3, 3–6, that the excess l_z of the observed value z_1 over the computed value z' of 5 6 is given by

$$- (C_1) + (C_2) - (C_1) + (C_2) - l_z \qquad (1)$$

where (C_1), (C_2),   denote the corrections to the angles C_1, C_2,   This is the azimuth condition equation.

In order that the correction arising from the azimuth equation may not disturb the conditions of closure existing among the angles of the triangles, it is necessary that the corrections to the angles should satisfy the conditions

$$(A_1) + (B_1) + (C_1) \cdot \ 0$$
$$(A_2) + (B_2) + (C_2) - 0 \qquad (2)$$

The unknowns in equations 1, 2 are subject to the relation

$$(A_1)^2 + (B_1)^2 +   = \text{a min.}$$

Call k_1, k_2, k_3, k_4 the correlates of equations (2), and k the correlate of equation (1), and we have the correlate equations

$$k_1 \quad = (A_1) \quad k_2 \quad = (A_2) \quad k_3 \quad = (A_3) \quad k_4 \quad \cdot \cdot (A_4)$$
$$k_1 \quad = (B_1) \quad k_2 \quad = (B_2) \quad k_3 \quad = (B_3) \quad k_4 \quad (B_4)$$
$$k_1 - k = (C_1) \quad k_2 + k = (C_2) \quad k_3 - k = (C_3) \quad k_4 + k = (C_4)$$

---

* For methods of doing this see Lee's *Tables and Formulae*, Washington, 1873, Coast Survey Report, 1875

whence the normal equations

$$3k_1 \qquad\qquad\qquad - \; k = 0$$

$$3k_2 \qquad\qquad + \; k = 0$$

$$3k_3 \qquad - \; k = 0$$

$$3k_4 + \; k = 0$$

$$- k_1 + k_2 - k_3 + k_4 + 4k = l_z$$

If we had n triangles instead of four, the normal equations would be of similar form, and, solving, we should find

$$k = \frac{3l_z}{2n}$$

$$+ k_1 = - k_2 = + k_3 = \qquad = \frac{l_z}{2n}$$

and therefore the corrections

$$(A_1) = \quad \frac{1}{2n}\, l_z, \quad (A_2) = - \frac{1}{2n}\, l_z,$$

$$(B_1) = \quad \frac{1}{2n}\, l_z, \quad (B_2) = - \frac{1}{2n}\, l_z,$$

$$(C_1) = - \frac{1}{n}\, l_z, \quad (C_2) = + \frac{1}{n}\, l_z,$$

Hence the rule *Divide the excess of the observed over the computed azimuth by the number of triangles, and apply one-half of this quantity to each of the angles adjacent to the flank sides on one side of the chain, and the total quantity, with the sign changed, to the third angle   The signs are reversed for the angles on the other flank*

If the azimuth mark is not on a triangulation side the line of azimuth may be swung on to such a side by adding the angle between the mark and the side

Ex The sketch represents the secondary triangulation of Long Island Sound   The sides 1–2, 14–15 are the lines of junction with the primary sys-

Fig 46
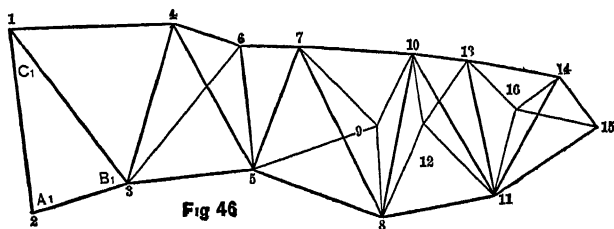
tem   These lines are assumed to remain unchanged in azimuth in the adjustment, and the secondary system is to be made to conform to them

The main chain of triangles joining the two primary lines is indicated in the figure by heavy lines   The number of these triangles is 11   The system has been adjusted for local and geometrical conditions, and the resulting angles of these triangles are as follows

Angle		log sin diff 1″	Sph excess	Angle		log sin diff 1″	Sph excess
A_1	82° 49′ 19″ 25	0 27		A_7	58° 25′ 08″ 08	1 29	
B_1	64° 55′ 40″ 32	0 99	3″ 66	B_7	82° 16′ 16″ 16	0 29	3″ 63
C_1	32° 15′ 04″ 09			C_7	39° 18′ 39″ 39		
A_2	52° 37′ 47″ 98	1 61		A_8	69° 12′ 06″ 02	0 80	
B_2	74° 10′ 50″ 56	0 60	4″ 96	B_8	71° 58′ 54″ 18	0 68	3″ 03
C_2	53° 11′ 26″ 42			C_8	38° 49′ 02″ 83		
A_3	69° 20′ 16″ 03	0 79		A_9	59° 18′ 26″ 33	1 25	
B_3	67° 43′ 43″ 33	0 86	3″ 53	B_9	102° 08′ 26″ 94	—0 45	1″ 33
C_3	42° 56′ 04″ 16			C_9	18° 33′ 08″ 05		
A_4	39° 31′ 29″ 66	2 55		A_{10}	67° 38′ 52″ 77	0 87	
B_4	120° 36′ 36″ 11	—1 25	1″ 31	B_{10}	70° 26′ 23″ 26	0 75	2″ 41
C_4	19° 51′ 55″ 54			C_{10}	41° 54′ 46″ 38		
A_5	83° 19′ 31″ 08	0 25		A_{11}	29° 43′ 55″ 99	3 69	
B_5	68° 57′ 47″ 75	0 81	1″ 42	B_{11}	55° 23′ 09″ 36	1 45	2″ 13
C_5	27° 42′ 42″ 58			C_{11}	94° 52′ 56″ 78		
A_6	87° 25′ 26″ 66	0 09					
B_6	44° 10′ 45″ 25	2 17	3″ 47				
C_6	48° 23′ 51″ 56						

A geodetic computation for latitude, longitude and azimuth was carried through from the line 1–2 to 14–15, using the above angles   It was found that approximately

observed az of 14–15 — computed az of do = — 2″ 93

Hence the corrections to the angles of the triangles for this discrepancy in azimuth are for the

first triangle $(A_1) = -$ 0″ 13   second triangle $(A_2) = +$ 0″ 13
$\qquad\qquad (B_1) = -$ 0″ 13 $\qquad\qquad\qquad (B_2) = +$ 0″ 13
$\qquad\qquad (C_1) = +$ 0″ 26 $\qquad\qquad\qquad (C_2) = -$ 0″ 26

and so on

161 **Adjustment for Discrepancy in Bases.**—This is fully explained in Arts 168–170

Using the last form given in Art 170, we may write the base line equation

$$[\partial_A(A) - \partial_B(B)] = l$$

from which, using the values of A_1, B_1,   found in the azimuth adjustment as first approximations, further corrections to the angles are found, as in equations 5, Art 170 Since the angles C do not enter into this adjustment, the corrections resulting will not disturb the adjustment for azimuth already made

The advantage of this method of proceeding is that the subsequent work does not disturb any adjustment already made and thus render it necessary to make a new approximation   The labor is reduced to a minimum, and the results obtained are practically close enough

For an example see Ex 1, Art 170

162 **Adjustment for Discrepancy in Latitude and Longitude.**—The corrections to the angles arising from discrepancy in azimuth and in bases having been applied, the value of one base computed from another will agree with the measured value in direction and length

45

The discrepancy in position, as shown by the differences be-tween observed and computed latitudes and longitudes, alone remains   This discrepancy, being small, may be eliminated closely enough by distributing it proportionally from one end of the chain of triangles to the other, accord-ing to Bowditch's rule as given in Ex 5, Art 110—that is, the error in latitude in proportion to the longitudes, and the error in longitude in proportion to the latitudes of the several stations   Each station, being thus made slightly eccentric, is next reduced to centre, when the whole net will be consistent

CHAPTER VII

163 During the present century two forms of apparatus have been used in the measurement of primary bases, the compensation bars and the metallic-thermometer apparatus On the English Ordnance Survey the two principal lines, the Lough Foyle and Salisbury Plain bases, were measured with the Colby compensation bars Most of the bases of the U S Coast Survey and five of the eight bases of the U S Lake Survey were measured with the Bache-Wurdemann compensation apparatus On the Continent of Europe the Bessel metallic-thermometer apparatus is very generally used

Indications are not wanting that both forms will be supplanted before long by an apparatus consisting of simply a single metallic bar *

The essential part of any form of base apparatus consists of one or two bars of metal, usually from 4 to 6 metres in length and of about 40 × 15 mm cross-section If the extreme points of a bar are the limits of measure, so that a measurement is made by contact [end-measures], two bars are necessary If, however, the length of a bar is considered to lie between two marks made on it [line-measures], so that in a measurement the transition from one bar to the next depends, not on the stability of the bar, but on some outside appliance, only one bar is necessary

Descriptions of the various forms of apparatus used on different surveys will be found in reports of those surveys

* See Ibanez, *Zeitschr für Instrumentenkunde*, 1881   Also Art 166

164 **Precision of a Base-Line Measurement.**—For clearness it will be necessary to outline the principles on which the measurement is made

First, we must find the length of the measuring bar in terms of some standard of length , and as the measurements of the line itself are made at various temperatures, the co-efficients of expansion of the metals in the measuring apparatus must also be known   Comparisons must, therefore, be made with the standard during wide ranges of temperature, and as these comparisons are fallible, the results found for length and expansion will be more or less erroneous

The principle involved in the measurement is exactly the same as in common chaining with chain and pins. There are, indeed, various contrivances for getting a precision not looked for in chaining, such as for aligning the measuring bar, for finding the inclination of each position of the bar, and for establishing fixed points for stopping at and starting from in measurement   But these make no change in the essential principle

The errors in the value of a base line may, therefore, be considered to arise from two principal sources, comparisons and measurement   Experience has shown that the main error arises from the comparisons, and that, even if our modes of measuring the base line itself were perfect, the precision of the final value would be but little increased so long as the methods of comparison are in their present state   Thus in the Lake Survey primary bases, if the field work had been without error, the total p e of the bases would have been diminished only about $\frac{1}{45}$ part

These errors differ essentially in character   An error arising from the comparisons, being the same for each bar measurement, is cumulative for the whole base, while errors arising in the measurement of the base itself, were the measurements repeated often enough and the conditions sufficiently varied, would tend to mutually balance, and could, therefore, be treated by the strict principles of least

squares   But as the number of measurements is not often more than 2 or 3, and as these are made usually at about the same season of the year, only a comparatively rough estimate of the precision is to be looked for

As a check on the field work a base is usually divided into sections by setting stones firmly in the ground at approximately equal intervals along the line, so that instead of being able to compare results at the end points only, we may compare results just as well at 6 or 8 points In this way a better idea of the precision of the work is obtained, as we have 6 or 8 short bases to deal with instead of a single long one

We proceed now with the problem of determining the precision of measurement   It may be stated as follows A base is measured in n sections with a bar of a certain length, each section being measured n_1 times   By the first measurement the first section contains M_1' bars, the second M_2' bars,   , by the second measurement the first section contains M_1'' bars, the second M_2'' bars,   , and so on.   The weights of the measurements in order being $p_1', p_2', \ldots, p_1'', p_2'', \ldots,$ respectively, required the m. s. e of the most probable value of the base

Let V_1 most prob value of first section

V_2 most prob value of second section

·   ·   ·

then we have the observation equations

First section,	V_1	M_1'	v_1'	wt	p_1'
	V_1	M_1''	v_1''	wt	p_1''
	·	·			·
Second section,	V_2	M_2'	v_2'	wt	p_2'
	V_2	M_2''	v_2''	wt	p_2''
	·	·	·		

and so on.

Now, either of two assumptions may be made

(a) In the first place, that the precision of the measurement of each bar is the same throughout the different sections

We have, then, nn_1 equations containing n unknowns, and the normal equations are

$$+ [p_1]V_1 \qquad = [p_1 M_1]$$
$$\qquad + [p_2]V_2 = [p_2 M_2]$$

whence V_1, V_2,   are known, and therefore the whole line $V = V_1 + V_2 +$ $+ V_n$ is known

The mean-square error μ of an observation of weight unity—that is, of a single measurement of a bar—is given by (see Art 99)

$$\mu = \sqrt{\frac{[pvv]}{\text{No of obs} - \text{No of indep unknowns}}}$$

$$= \sqrt{\frac{[pvv]}{n(n_1 - 1)}}$$

Now, the length of the measuring bar being, taken as the unit of measurement, the weight of a section, as depending on the measurement, may be expressed in terms of the number of bars measured

For since μ is the m s e of a measurement of a single bar, the m s e of the measurement of a length of M bars is $\mu \sqrt{M}$ Hence $\frac{1}{M}$ is the weight of a measurement of length M when the weight of a measurement of the unit of length is unity

Writing, therefore, for the weights p their values in terms of M,

$$\mu = \sqrt{\frac{1}{n(n_1 - 1)}\left[\frac{vv}{M}\right]}$$

In the case usually occurring in practice, where the line is measured twice, we may put this formula in a form more

convenient for computation   For if the first measurement
of the n_1 sections gave lengths M_1, M_2,   , and the second
measurements gave lengths $M_1 + d_1$, $M_2 + d_2$,   for the
same sections in order, then, since

$$[vv] = \left[\left(+\frac{d}{2}\right)^2 + \left(-\frac{d}{2}\right)^2\right] = \frac{[dd]}{2}$$

we have for the m s e of one measurement of a bar and for
the mean of two measurements respectively

$$\sqrt{\frac{1}{2n}\left[\frac{dd}{M}\right]} \text{ and } \tfrac{1}{2}\sqrt{\frac{1}{n}\left[\frac{dd}{M}\right]}$$

Hence the m s e of a single measurement and of the mean
of the two values of the whole base are

$$\sqrt{\frac{[M]}{2n}\left[\frac{dd}{M}\right]} \text{ and } \tfrac{1}{2}\sqrt{\frac{[M]}{n}\left[\frac{dd}{M}\right]}$$

the number of bars in the line being $[M]$

Ex The Bonn Base, measured in 1847, near Bonn, Germany, with the
original Bessel metallic-thermometer apparatus   The base was a broken
one, the two parts making an angle of 179° 23′   Each part was measured
twice, as follows *

	Differences	No of bars
	L	
Northern Part, Sec 1	− 0 183	116
Sec 2	+ 0 091	87
Sec 3	− 0 013	61
		—— 264
Southern Part, Sec 1	− 0 007	92
Sec 2	+ 0 095	60
Sec 3	+ 0 757	131
		—— 283

* *Das rheinische Dreieilsnetz*   Berlin, 1876

Hence the m s e of the northern part, arising from errors of no 1 ... to 10 ... 4 only, is

$$\tfrac{1}{6}\sqrt{\dfrac{264}{3}\left(\dfrac{183'}{116}+\dfrac{004'}{87}+\dfrac{013}{01}\right)} \quad \pm \mathrm{''}...$$

and the m s e of the southern part is

$$\tfrac{1}{6}\sqrt{\dfrac{283}{3}\left(\dfrac{007'}{92}+\dfrac{005'}{60}+\dfrac{7\,\mathrm{v}'}{131}\right)} \quad \pm \mathrm{''}...$$

The other two main sources of error are

1 Error in comparison of the measuring bars with one another

2 Error in the determination of their length

The m s e arising from these sources are respectively

$$\pm\ 0\ 386, \quad \pm\ 0\ 313\ \text{for the northern part}$$
$$\pm\ 0\ 391, \quad \pm\ 0\ 335\ \text{for the southern part}$$

Remembering that these latter errors are systematic, we have, finally

$$\text{m s e of base} = \sqrt{003'+327'+(350+691)+(313+...)}$$
$$\pm\ 07$$

(b) In the second place, if we assume that the law of precision of the measurements of the different sections is unknown, and that these sections are independent, we have for the mean of the values of the several sections and their m s e

$$V_1 = \dfrac{[p_1 M_1]}{[p_1]}$$
$$\mu_{V_1}^{2} = \dfrac{[p_1 v_1 v_1]}{[p_1](n_1-1)} = \dfrac{[v_1 v_1]}{n_1(n_1-1)} \quad \text{since } p_1' = p_1'' \quad . \qquad \tfrac{1}{17}$$

$$V_2 = \dfrac{[p_2 M_2]}{[p_2]}$$
$$\mu_{V_2}^{2} = \dfrac{[p_2 v_2 v_2]}{[p_2](n_1-1)} = \dfrac{[v_2 v_2]}{n_1(n_1-1)} \quad \text{since } p_2' = p_2' \quad .. \qquad \tfrac{1}{17}$$

If l denotes the whole line, so that

$$l = l_1 + l_2 + \ldots + l_n$$

then, since the measurements are independent,

$$\mu^2 = \mu_1^2 + \mu_2^2 + \ldots$$

$$= n_1 n_1 \; \mu_1^2 \left([\epsilon_1 \epsilon_1] + [\epsilon_2 \epsilon_2] + \ldots \right)$$

and the (m. s. e.) of a single measurement of the line

$$\mu_i = \frac{1}{n_1} \left([\epsilon_1 \epsilon_1] + [\epsilon_2 \epsilon_2] + \ldots \right)$$

The number of bars in the line being $[M]$, we have for the average value of the (m. s. e.) of a single measurement of a bar

$$\frac{1}{n} = \frac{[\epsilon_1 \epsilon_1] + [\epsilon_2 \epsilon_2] + \ldots}{[M]}$$

If, for example, the line has been measured twice, and $d_1, d_2, \ldots, d_i$ are the differences of the measurements of the several sections, then

$$[\epsilon_1 \epsilon_1] = \frac{d_1^2}{2}$$

$$[\epsilon_2 \epsilon_2] = \frac{d_2^2}{2}$$

$$\ldots$$

and therefore

$$\mu_i^2 = \frac{[d^2]}{4}$$

and the (m. s. e.) of a single measurement of a bar is $\frac{1}{2}\dfrac{[d^2]}{[M]}$.

16

Ex The Chicago Base, measured in 1877 with the Repsold metallic-thermometer apparatus belonging to the U S Engineers The base was divided into 8 sections and was measured twice

Section	No of Bars	Diff of Measures
		mm
I	227 25	— 1 3
II	230 25	+ 2 5
III	234 50	+ 2 3
IV	232	+ 0 7
V	231	+ 1 5
VI	225	+ 1 1
VII	300 50	+ 1 3
VIII	196 80	— 0 2

Taking the errors of the different sections as independent, the p e of the mean of the two measures of the base is

$$0 6745 \sqrt{\frac{[d']}{4}} = \overset{mm}{1} 46$$

The p e arising from the other sources of error were

<div style="margin-left:2em">

mm

(1) measuring bar,   + 6 38
(2) metallic thermometer,   + 2 82
(3) elevation above mean tide, N Y , + 0 36

</div>

Assuming these to be independent, the p e of the Chicago Base at sea level is

$$\sqrt{1 46^2 + 6 38^2 + 2 82^2 + 0 36^2} = \overset{mm}{7} 14$$

Of the two assumptions (a) and (b), the first is due to Bessel and the second to Andræ For a more elaborate discussion of the points involved see *Astron Nachr.*, Nos 1924, 1935, and *V J Schr d Astr Ges* , 1878, p 69

165 Length of Base and Number of Bases necessary in a Triangulation.—In computing a side of a triangle from another and shorter side as base, the loss of precision arising from the influence of the acute angle opposite to the base is the greater the more acute that angle is To guard against this, if a measured base were a side of a triangle it would be necessary that its length be

about the average length of the triangle sides of the system. In the older work the practice was to measure long bases, even as long as 10 kilometres and over. The modern practice was introduced by Prof. Schwerd, of Speyer, Germany, in 1822. By measuring a short base of 860^m and checking on to a line computed from a base of 15 kilometres, he found a result agreeing within 0^m 1. His conclusion was that "with a small expenditure of time, trouble, and expense the base of a large triangulation can be determined by a small exactly measured line." [x]

A similar conclusion was reached by Gen. Ibanez as the result of measuring the base of Madridejos, Spain, in 1859 He says: "The question, so much disputed between French and German geometers, as to whether it is necessary to measure long bases, or if short bases are sufficient, had occupied the attention of the observers. Taking advantage, therefore, of the favorable opportunity which presented itself to them, they proposed to compare the results obtained from the direct measure of the whole base with those calculated from a special triangulation depending on the central section of the base." [†]

The adjustment of this triangulation net gave rise to 36 angle equations and 48 side equations containing 90 unknown quantities. The net is shown in the figure

"We give in the following table the results of the direct measure reduced to the sea level, and their comparison with the values found trigonometrically.

Measured	Triangulation	Fig.47
''	*''''*	
457? 10	457? 103	
1 116. 161	1 116. 168	
1 141 1041		
1, 141 1'1	1? 141 122	
62 1 11	887? 611	
1 966 1.'21 5	1 966? 889	

* ... , ... Speyer, 18 2   † Astron. Nachr., No 1461

" The remarkable agreement which the two operations present is sufficient authority to limit the length of bases, and to be satisfied with those of 2 or 3 kilometres in length, always on the condition that they are joined to the sides of the main triangulation by means of a system of lines arranged so as to be able to apply to it the proper method of adjustment "

If errors arising from the angles of the triangles are neglected, it is easy to show that if a short base M is measured n times, and the arithmetic mean of the n measurements taken as base from which a line equal to nM is derived by triangulation, the precision of this line is the same as if it had been measured once directly *

For if $\mu_M =$ the m s e of a length M, then $\mu_M \sqrt{n}$ is the m s e of a length nM Also, the m s e of the arithmetic mean of n measurements of the line M is $\dfrac{\mu_M}{\sqrt{n}}$, and when from this line the line nM is derived trigonometrically its m s e is $\dfrac{\mu_M}{\sqrt{n}} n = \mu_M \sqrt{n}$, agreeing with the preceding

Since, however, the principal errors arise from the triangulation, the advantage is 'with the longer base

We may estimate the relative amount of influence of errors in the base and of errors in the angles in a chain of triangles, on the value of any side computed from the base, as follows

Assuming the triangles to be approximately equilateral, we have (Ex 9, Art III)

$$\mu_{a_n}{}^2 = \mu_b{}^2 + \tfrac{2}{3}\mu^2 \sin^2 1'' \, b^2 n$$

where a_n is the computed side, b the base, n the number of triangles intervening, and μ the m s e of a measured angle

Suppose, for the sake of fixing our ideas, that $b = 10000^m$, then

$$u_{a_n}{}^2 = \mu_b{}^2 + 1600 \, n\mu^2 \text{ millimetres}$$

* *Gradmessung in Ostpreussen*, p 36

Now, with a primary base apparatus a precision of a m. s. e. of 1 mm in 1000 m can be easily reached.[*]   Hence from the above formula we see that even in a short chain of triangles the error of a side arising from the error of the base is small in comparison with that arising from the errors in the angles measured.

Also, since the m. s. e. of a side arising from errors in the angles measured increases as the square root of the number of triangles from the base (Ex. 9, Art. 111), we conclude that it is better to measure bases frequently with moderate precision rather than to measure a few at long intervals in the net, but with great precision.

But little is gained in precision by repeating the measurement of the base many times.   We have seen in Art. 104 that the main sources of error to be feared arise from the comparisons with the standards and are independent of the measurement of the line.   Accordingly, though bases have in the present century been measured from 6 to 8 times, it is the general custom now to do so only twice. In this way any gross error is checked and a sufficiently close precision determination of the measurement can be found.   As a compromise between the error arising from the connection with the triangulation and the systematic error introduced by the base apparatus itself, the general practice is to measure bases of from 4 to 8 kilometres [in length, or about one-sixth of the triangle sides.   Besides, long bases are not always to be had in positions just where wanted as triangle sides, as the configuration of most countries will not allow of it.

166. Measuring a base line is not necessarily a difficult operation. The great trouble with all ... hitherto employed arises from temperature ... But with the apparatus proposed by Mr. ... W ... in which the measuring bar is simply ... melting ice, this difficulty would be ... The length of the bar would remain ... the measurement, as its temperature ... that of melting ice. The same temperature ... time be had at which to find the length of the bar ... of the official standard of length.

The apparatus might be constructed ... the measuring bar, a bar of steel ... mm ... length, placed in a circular ... steel tube ... made stiff by bracing, but as light ... top of this tube slots of about ... in width ... to allow the introduction of ice around ... for drainage would be at the centre of the ... under side. For support during the ... trestles placed ... in from the ends would ... of flexure would be got rid of by having the ... marks showing the length of the bar placed on the ... axis of the bar. The reading microscope ... ratus, sector and level for determining the ... bar during measurement, such as there ... for the U. S. Engineers. The mode of ... same as with the Repsold apparatus ... computation necessary to reduce the measurement ... in this way would be small in comparison with that required with the forms of apparatus at present ... use.

167. **Connection of a Base with the Main Triangulation.** We shall now consider the ... of connecting a base line with a side of the ... with the least possible loss of precision. The ...

** See *Professional Paper* ...
gold base apparatus.

geodetic questions, is not to be decided by least-squares'
method alone.

There are two points to be considered—as little loss of
precision as possible, combined with economy of work in the
measurement and reduction of the intervening triangula-
tion. To secure the first we must use only well-shaped
triangles, avoiding in particular very acute angles opposite
the bases. To secure the second as few stations as possible
should be occupied.

The solution is a tentative one. Various forms of con-
nection may be tried, and the m. s. e. of the triangle sides
computed from the bases by the methods of Chapter V., on
the hypotheses of a regularity of figure which conforms
more or less closely to the case in hand. Many of the re-
sults will be found in Exs. 10-15, Art. 111. These results we
shall now make use of.

Taking the length of a triangulation side to be from 15
to 30 miles, as giving the best results in angle measurements,
the proper length of the base would be from 3 to 5 miles—
that is, about one sixth of a triangle side. Now, if the con-
nection between the base AB and a triangle side 6 times
the base is through a chain of equilateral triangles of the
form of Fig. 14, or of the rhomboidal form of Fig. 15, com-
posed of two similar triangles, in which the first diagonal
AB is 6 times the base, or of the rhomboidal form of Fig. 16,
in which the second diagonal is 6 times the base, the re-
spective weights of the derived side are, roughly, as the
numbers 10, 4, 7.

Hence, so far as the precision is concerned, the advantage
is with the first form. When, however, we consider that
this form requires 11 stations to be occupied, while the third
need only 6, we see that when amount of labor as well as
degree of precision is taken into account the advantage is
with the latter form. The second form, though the most
economical, is condemned by the acute angles that occur in
it opposite to the base.

Similar comparisons of other forms of connection will

lead us to the conclusion that the rhomboidal form (Fig 16), in which a side equal in length to about 6 times the base can be found by passing through two sets of similar triangles from the base, is the normal form, and that the forms used in practice should approximate to it as closely as the nature of the country will allow   As examples take the connections of the Coast Survey Atlanta Base (1872- 1873), the Lake Survey Chicago Base (1877), and the Prussian Gottingen Base (1880)

Fig 50

Fig 49

Fig 48

ATLANTA   CHICAGO   GOTTINGEN
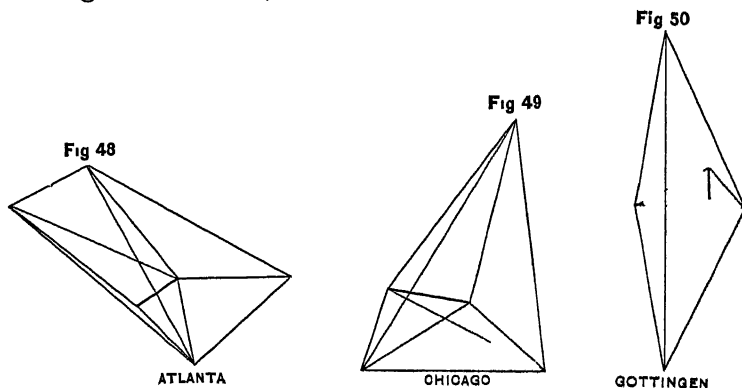
The form of the triangulation net to connect the base with the main triangulation being decided on, it becomes a question (but only a secondary one) to decide with what care the several angles in this net should be measured in order to attain the greatest precision with a given expenditure of labor   This will be found fully discussed in *Zeitschr. fur Vermess* , 1882, pp 122 seq

168 **Adjustment of a Triangulation when more than one Base is considered.**—In all discussions so far we have considered only one measured base   But in a net of triangles, bases must be measured at intervals, for reasons already assigned   In computing an intermediate side from different bases discrepancies will be found   How shall these discrepancies be treated?

The triangulation adjustment could have been made in

such a way that no discrepancy would show itself in passing from one measured base to another. It was only necessary to introduce equations connecting the bases in the form of an ordinary side equation, thus (Fig. 1?),

$$\frac{a}{b} = \frac{\sin A_1 \sin A}{\sin B_1 \sin B_1}$$

where a, b are two bases, and A_1, B_0, . . . are the angles of continuation.

There is no objection to doing this so long as the bases have been measured with the same apparatus and the intervening triangulation is first rate. But as the discrepancy between the measured value of the base and its value as computed from another base through the triangulation affords a good test of the quality of the work, it is better not to introduce an equation connecting the bases into the first adjustment.

165. Suppose that the triangulation has been adjusted in sections, each with reference to a single measured base, as explained in Chapter VI., and we wish to adjust for the discrepancy arising from computing one base from another through a chain of the best shaped triangles in the system. It will be sufficient to confine our attention to this chain of triangles, all tie lines being rejected.

(1) *Rigorous Solution.* The condition equation to be satisfied, arising from the connection of the bases, may be written

$$\frac{a + (a)}{b + (b)} = \frac{\sin \frac{1}{2}A_1 + (A_1)\{\sin \frac{1}{2}A_1 + (A_0)\}}{\sin \frac{1}{2}B_1 + (B_1)\{\sin \frac{1}{2}B_1 + (B_0)\}} \cdot$$

where a, b, A_1, B_0 . . . are measured values, and (a), (b), (A_1), (B_0), . . . are their most probable corrections.

Taking logs and reducing to the linear form (Ex. 4, Art. 64),

$$\delta_1(a) + \delta_2(b) + [\delta_4(A) - \delta_6(B)] = l$$

where l is the excess of the observed over the computed value of log a, and δ_a, δ_b, δ_A, δ_B are the log differences as usual

Also, since the angles of each triangle in the chain must satisfy the condition of closure, we have the conditions

$$(A_1)+(B_1)+(C_1)=0$$
$$(A_2)+(B_2)+(C_2)=0$$
$$\cdot \qquad \cdot$$

with

$$\frac{(a)^2}{\mu_a^2}+\frac{(b)^2}{\mu_b^2}+\left[\frac{1}{\mu^2}\left\{(A)^2+(B)^2+(C)^2\right\}\right]=\text{a min}$$

where μ_a, μ_b are the m s e of the bases, and μ_1, μ_2, the m s e of the three angles of each of the triangles in the chain

The solution may be carried out by the method of correlates

If the bases have been measured with different apparatus the question of the comparison of the measuring bars with the standards of length is the important one After that has been satisfactorily settled the connection of the bases may be treated as above

Ex This mode of reduction may be illustrated by the simple case of a triangle having two sides and the three angles measured

In the triangle W Base, E Angle, W Angle (*1BC*), Sandusky Base, there were measured

$$BC = 6742\ 420 \overset{ft}{\pm} 0\ 010$$
$$AC = 6602\ 386 \overset{ft}{\pm} 0\ 010$$

From the adjustment of the triangulation with reference to one measured base, and which was carried out by the methods already explained in Chapter VI , the three adjusted angles of the triangle were found to be

$$BAC = \quad 1° \ 07' \ 51'' \ 35 \pm 0'' \ 20$$
$$ABC = \quad 1° \ 06' \ 26'' \ 74 \pm 0'' \ 20$$
$$ACB = 177° \ 45' \ 41'' \ 91 \pm 0'' \ 20$$

Required the most probable values of the sides and angles of the triangle from a second adjustment into which the two measured sides enter

Thus, if ds, (B), (C), (c), etc., are the corrections to the above quantities in order, the condition equations are

$$(A) + (B) + (C) = 0$$

$$c + d + s(c + c) = \sin \tfrac{1}{2}(r + o)' \, sr = s + (B)\{$$
$$(c c) + (d c + c) = \sin \tfrac{1}{2}(r + o)\, o = o + (B)\{$$

or, reducing the latter to the linear form,

$$o + d(r) + o t s(b) = r + o q(A) + r + o b y(B) = o \cdot o t t$$

with

$$\frac{(o)}{\{ o r \}} + \frac{c t}{\{ o r \}^2} + \frac{(H)'}{(20)^2} + \frac{(B)'}{(20)^2} + \frac{(C)'}{(20)^2} = \text{a min}$$

that is

$$pq(r r) + pq q o' + (b)^2 + (B)^2 + (C)^2 = \text{a min}$$

The solution of the o equations gives

$$
\begin{array}{lll}
(r) & + \, 0 \cdot xxxx & (A) = o + o \cdot \\
( ) & 0 \cdot xxxx, & (B) = + o + o \cdot \\
& & (C) = o' \cdot oo
\end{array}
$$

This example is noteworthy as showing the combination of heterogeneous measures on the same minimum equation.

(8) *Approximate Solutions.* The two of most importance are (a) when the angles alone are changed, (b) when the bases alone are changed. Either of these is practically more important than the rigid solution, as a base line in good work receives a very small correction from the adjustment.

(a) When the angles alone are changed. The formulas to be used in this case follow from those of the rigorous method just given by putting the base corrections equal to zero. Thus, if all of the adjusted angles are of the same weight, then, since $(a) = o$, $(b) = o$, the base line equation becomes

$$[\delta_A(A) - \delta_B(B)] = l \qquad (1)$$

with

$$[(A)^2 + (B)^2 + (C)^2] = \text{a min}.$$

The angle equations are as before.

Hence if k is the correlate of the base-line equation, we have, by eliminating the angle equation correlates,

$$
\begin{aligned}
(A_1) &= \quad\ (2\delta_A' + \ \delta_B')k \\
(B_1) &= -\ \ (\delta_A' + 2\delta_B')k \\
(C_1) &= -\ \ (\delta_A' - \ \delta_B')k \\
\\
(A_2) &= \quad\ (2\delta_A'' + \ \delta_B'')k \\
(B_2) &= -\ \ (\delta_A'' + 2\delta_B'')k \\
(C_2) &= -\ \ (\delta_A'' - \ \delta_B'')k
\end{aligned}
\qquad (2)
$$

$$\bullet \qquad \bullet \qquad\qquad \bullet$$

whence, by substituting in the base-line equation,

$$k = \frac{l}{2[\delta_A'^2 + \delta_A\,\delta_B + \delta_B'^2]}$$

Hence the corrections to the angles are known

A still further approximation may be made   It is evident that the corrections to the angles C are small compared with those to A and B, and that they vanish when $A = B$. Hence, as we have assumed the triangles to be well shaped, we may take

$$(C_1) = (C_2) = \qquad = 0 \qquad (3)$$

The angle equations then become

$$
\begin{aligned}
(A_1) + (B_1) &= 0 \\
(A_2) + (B_2) &= 0
\end{aligned}
\qquad (4)
$$

whence

$$(A_1) = -(B_1) = \frac{\delta_A' + \delta_B'}{2}\, k$$

$$(A_2) = -(B_2) = \frac{\delta_A'' + \delta_B''}{2}\, k \qquad (5)$$

$$\bullet \qquad\qquad\qquad \bullet$$

where

$$k = \frac{2l}{[(\delta_A + \delta_B)^2]}$$

Ex 1 To find the changes in the angles resulting from the equation connecting the lengths of the lines 1–2 and 14–15 in the triangulation of Long Island Sound (Fig 46)

The excess of the log of the observed value of 14–15 over the value computed from 1–2 through the triangulation is 4 00 in units of the sixth decimal place

If the lines themselves receive no correction the condition equation, expressed in the linear form, is (see table, p 346)

$$0\ 27(A_1) - 0\ 99(B_1) + 1\ 61(A_2) - 0\ 60(B_2) + \quad = 4\ 00$$

Now,

$$[(\delta_A + \delta_B)^2] = 51\ 48$$

and therefore

$$(A_1) = -(B_1) = \frac{0\ 27 + 0\ 99}{51\ 48} \times 4\ 00 = 0''\ 10$$

$$(A_2) = -(B_2) = \frac{1\ 61 + 0\ 60}{51\ 48} \times 4\ 00 = 0''\ 17$$

$\cdot \quad \cdot \qquad\qquad \cdot$

the corrections required

Ex 2 To find the precision of a side in a chain of triangles joining two bases

Having found the adjusted angles, we may find the weight of the log of any side of continuation as the mth in a chain of n triangles joining two bases, all of the angles being of the same weight

For simplicity in writing take $m = 2$, $n = 3$

The condition equations are

$$(A_1) + (B_1) + (C_1) = 0$$
$$(A_2) + (B_2) + (C_2) = 0$$
$$(A_3) + (B_3) + (C_3) = 0$$

$$\delta_A'(A_1) - \delta_B'(B_1) + \delta_A''(A_2) - \delta_B''(B_2) + \delta_A'''(A_1) - \delta_B'''(B_1) - l$$

and the function F, whose weight is to be found,

$$F = \delta_A'(A_1) - \delta_B'(B_1) + \delta_A''(A_2) - \delta_B''(B_2)$$

The solution follows at once from Eq 15, Art 111   The result is

$$u_F = \frac{2}{3} [\delta_A^2 + \delta_A\ \delta_B + \delta_B^2]_1^2 \frac{\delta_A'''^2 + \delta_A'''\ \delta_B''' + \delta_B'''^2}{[\delta_A^2 + \delta_A\ \delta_B + \delta_B^2]_1^3}$$

the summation of the δ's extending between the limits indicated

Let b_1, l_1 ... be the measured lengths of the bases, and p_1, p_2 ... their weights

If s is the most probable length of a common side computed from the bases, and l_1, l_2 ... are the ratios of the lengths of the bases to this side, then

$$s = \frac{b_1}{l_1} \qquad \text{weight } p \, l_1, \text{ and weight of log } \frac{b_1}{l_1} \text{ is } p_1 b_1{}^2$$

$$s = \frac{b_2}{l_2} \qquad " \qquad p \, l_2 \qquad " \qquad \log \frac{b_2}{l_2} \text{ is } p_2 b_2{}^2$$

. . . .

The weighted mean value of log s is

$$\frac{p_1 b_1{}^2 \log \frac{b_1}{l_1} + p_2 b_2{}^2 \log \frac{b_2}{l_2} + \cdots}{p_1 b_1{}^2 + p_2 b_2{}^2 + \cdots}$$

With this value of log s, leaving the angles of the triangles as they are, the sides may be recomputed, when, of course, the bases will be changed

We have seen in Art. 164 that in base line measurements the error arising from errors of comparison of the measuring bars is the main one, and that the error arising from the measurement of the line itself is relatively small   Hence we may take the m. s. e. of a base as proportional to the length, and the weight, therefore, inversely as the square of the length   The weight of each computed value of log s will then be the same, and we have now

$$\log s = \frac{1}{n}\left[\log \frac{b}{l}\right]$$

As the weights taken are tolerably close, this simple formula will give a result near enough for most work.

It is evident that if we compute each base from all of the others, and take the mean of the logarithmic values found, consistent values of the sides throughout the triangulation will be found by starting from any base.

Thus in the Coast Survey example already referred to,

	Epping Base	Mass Base	Fire Island Base
Measured,	3 9403143 4	4 2387077 4	4 1479535 3
Computed from Epping,		4 2387115 5	4 1479556 3
" " Mass ,	3 9403075 8		4 1479518 4
" " Fire Id ,	3 9403122 4	4 2387094 6	
Means,	3 9403113 9	4 2387095 8	4 1479536 7

There will now be no contradiction in the values of any triangle side deduced from the different bases

CHAPTER VIII

APPLICATION TO LEVELLING.

170. The object of levelling is to find the difference of height of two or more points on the surface of the earth. As only differences of height, and not absolute values, are required, some one height may be selected as the standard of reference, and to it any value at will may be assigned. The common custom in geodetic work is to take what is called the mean level of the ocean as the standard height. To be able to refer to it at any time a fixed mark is established, say at New York, and readings are taken to the surface of the ocean at high and low water for a lunation at least. The height of the fixed mark above mean tide is computed from these readings. Then all that is to be done in order to find the elevation of any point above mean tide is to run a line of levels between this point and the fixed mark just described.

There are two methods of levelling in common use in geodetic work—spirit levelling of precision and trigonometric levelling. The difference between the two consists mainly in the greater lengths of lines sighted over in the latter case.

Spirit Levelling of Precision.

171. For descriptions of instruments employed see Nivellement de Précision de la Suisse, par Hirsch et Plantamour, Genève et Bâle, 1867, seq.; Report Chief of Engineers U. S. A., 1886, App. OO; Report U. S. Coast Survey, 1876 App. 13, 1880 App. 11; Professional Papers Corps of Engineers U. S. A., No. 24, Chap. XXII.

The method of observing used on the Coast Survey is as follows   Two levelling rods are used   "Two lines are run simultaneously, with the rods usually at different distances from the instrument, and to prevent the gradual accumulation of error supposed to be due to running constantly in one direction, alternate sections are run in opposite directions   Each station of the instrument, therefore, contains two backsights and two foresights   In stations 1, 3, 5, etc, rod A, backsight, is invariably read first, then rod B, backsight, then rod A, foresight, and finally rod B, foresight   In stations 2, 4, 6, rod B takes precedence of A in both back and fore sights "

An important source of error in spirit levelling, and one very commonly overlooked, is the change in the length of the levelling rod from variations of temperature   From experiments made by the Prussian Land Survey, in which the rods were compared daily with a steel standard, the following fluctuations in length were found for four rods made of seasoned fir   *

Rod 13, from May 19 to Aug 18, 0 51 mm per metre
    14,   "   "   20 "   "   15, 0 46   "   "
     9,   "   "   24 " Sept   6, 0 37   "   "
   10,   "   "   24 "   "   6, 0 43   "   "

It is quite possible that errors from this source may largely exceed the errors arising from the levelling itself   Each field party should therefore be provided with the means of making a daily comparison of the rods used with a standard of length   A steel metre and a micrometer microscope mounted on a stand would be all that would be necessary

It is a curious fact in spirit levelling, but one abundantly verified, that a much greater discrepancy is to be expected in a duplicate line of levels if the lines have been levelled over in opposite directions than if both have been run in the same direction   This was noticed long since in a line

* *Nivellements der Landesaufnahme*, vol v   Berlin, 1883

of levels run under the direction of a committee of the British Association from the Bristol Channel to the English Channel (Portishead to Axmouth), a distance of 74 miles The increase in the discrepancy of the forward and backward levels was continuous Thus at 12 miles from Portishead it was 0 35 ft , at 23 miles 0 53 ft , at 37 miles 0 70 ft , at 49 miles 0 82 ft , at 59 miles 0 92 ft , and at 74 miles 1 03 ft [x]

In the winter of 1878–1879 a line of levels was run, under the direction of the U S Engineers, from Austin, Miss , to Friar's Point, Miss , a distance of about 44 kilometres " All lines were levelled in duplicate and in opposite directions When using two rods both were used on the same line, rod No 2 being used for the first, third,   backsight and for the second, fourth,   foresight, and rod No 3 for the second, fourth,   backsight and for the first, third,   foresight, always using rod No 2 on the closing bench mark " The results for the principal bench marks are given in the following table, where it will be seen that the discrepancy in the forward and backward levels increases from one end of the line to the other †

Distance	Bench Mark	Discrepancy
	Austin I	
16505 [m]	Trotter's Landing	+ 16 8 [mm]
4464	Glendale	+ 16 8
15950	Delta	+ 23 6
6602	Friar's Point I	+ 36 5

Experience on the survey of the great lakes and of the Mississippi River, and also on the survey of India, has shown that the personal peculiarities of the observer enter

* *Report British Association*, 1838   † *Report Chief of Engineers U S A* , 1879, p 1944

largely into levelling work   It would seem that a much more reliable result will be obtained if the line between two points whose difference of height is required has been run over not only in opposite directions, but in opposite directions by the same observer   The personal bias is probably due most largely to the fact that, the ends of the bubble not being sharply defined, different observers estimate the positions of these end points differently   To eliminate its effect completely the line should be levelled in opposite directions by each of a large number of observers

Again, as both rod and bubble should be read simultaneously at each foresight and at each backsight, and as this is physically impossible for one observer, the effect of the unequal heating of the instrument by the sun, even when shaded by an umbrella, has a tendency to cause a change in the position of the bubble in the interval between the readings of the rod and bubble   This error is cumulative, and its reduction to a minimum depends upon the skill of the observer

The rule adopted by the European Gradmessung for allowable discrepancy in a duplicate line of levels is that the p e of the difference in height of two points one kilometre apart should in general not exceed 3 mm , and should in no case exceed 5 mm   On the U S Coast Survey, for short distances a discrepancy between two levellings of a distance of D kilometres of an amount not exceeding $5\sqrt{2D}$ mm is allowed

172 **Precision of a Line of Levels.** --The precision of a line of levels will be given by the m. s e of a single levelling of a unit of distance, which we shall take to be one kilometre   The problem is quite analogous to that already discussed in Art 164, the unit of distance there being the length of a measuring bar

If we suppose that in running a single line of levels between two bench marks, A and B, the ground is equally favorable throughout, and that none but accidental errors may be expected to enter, then if μ is the m s. e. of the

unit of distance (one kilometre), the m. s. e. of a distance of D kilometres would be

$$\mu \sqrt{D}$$

in other words, *the weight of a levelling of a distance of D kilometres is will be inversely proportional to that distance*

Strictly speaking, we should find the weight from the m. s. e. μ, μ_1 . . . arising from all sources of error that enter into the work, such as from the comparisons of the levelling rod with the standard, from the nature of the country levelled over, from effects of change of length of sight, etc. Then, considering the errors arising from these sources to be independent, we should find finally the m. s. e. of each line levelled over to be $\sqrt{[\mu^2]}$, and the weight of the line would therefore be as $\dfrac{1}{[\mu^2]}$

The uncertainty attendant on estimating these sources of error is so great, and the influence of the distance so largely exceeds the influence of the others, that it is suffi cient to adopt the rule first given of weighting as the in verse distance It follows from this rule that better work is to be looked for if short sights are taken. Even with a first rate telescope sights should not be taken to exceed 100 metres.[*] As the rod is more easily read by the observer at short distances, the loss of time from the more frequent settings of the instrument is not so great as would at first appear.

Suppose now that in finding the difference of height of two bench marks, A and B, n, lines of levels have been run, and that the results have been compared at intermediate bench marks in succession D_1, D_2, . . . D_n kilometres apart,

---

[*] On the prairie for long sights these errors where the slope of the ground is steep the distance may be taken a great way profitably, and on comparatively level ground they may range from 100 to 300 metres, according to the condition of the weather and atmosphere.

The U. S. Engineers follow the rule " The length of sight will depend on the condition of the atmosphere, but the rod should deviate be no means high to be read distinctly. It will be seldom that length of sight exceeding 300 metres can be taken."

In the Prussian Land survey "the length of sight does rarely not been taken over 60 Only in precipitous terrain sight, in extreme stations is it permitted to exceed 40 , whereas a shorter sight is frequently taken.

then if $v_1', v_1'', \quad , v_2', v_2'', \quad ,$ denote the residual errors of the differences of height in the n sections, and $p_1', p_1'', \quad , p_2', p_2'', \quad ,$ denote the weights of the observed differences, we have, as in Art 164—

(1) On the hypothesis that the precision for each unit of distance (one kilometre) is the same throughout the different sections,

$$\mu^2 = \frac{[pvv]}{n(n_1 - 1)}$$

$$= \frac{1}{n(n_1 - 1)}\left[\frac{vv}{D}\right]$$

since the weights are inversely as the distances

(2) On the hypothesis of the independence of the sections between the intermediate bench marks, the average value of μ is given by

$$\mu^2 = \frac{n_1}{n_1 - 1}\left\{[v_1 v_1] + [v_2 v_2] + \quad \right\} \frac{1}{[D]}$$

When the number of levellings is two, and the observed differences of height at the several bench marks $D_1, D_2, \quad D_n$ kilometres apart give discrepancies of $d_1, d_2, \quad d_n$ respectively, then the above formulas reduce to

$$\frac{1}{2n}\left[\frac{d^2}{D}\right] \text{ and } \tfrac{1}{2}\frac{[d^2]}{[D]}$$

The m s e of a single levelling of the whole line would be respectively

$$\sqrt{\frac{[D]}{2n}\left[\frac{d^2}{D}\right]} \text{ and } \sqrt{\frac{[d^2]}{2}}$$

and for the m s e of the mean of the two measurements these values would each be divided by $\sqrt{2}$

Ex 1 In the precise Levelling (1880) of the U S Coast Survey on the Mississippi River two lines were run simultaneously between every two bench marks The following are the results from B M LIV to B M LV

B M	Distance	Rod *A*	Rod *B*	Difference
	ful	*m*	*m*	*mm*
	3 447	− 0 8702	− 0 8713	1 1
	3 806	+ 0 5805	+ 0 5738	6 7
	1 204	− 0 2154	− 0 2204	5 0
	3 038	+ 0 2667	+ 0 2664	0 6

It will be found that $\mu = 2^{mm}$ nearly

Ex 2 The distance AB has been levelled n different times   Calling d_2 the difference between the first and second measurements, d_3 the difference between the first and third, and so on, show that the m s e of an observation of weight unity is

$$\sqrt{\frac{n[d^2] - [d]^2}{n(n-1)}}$$

Ex 3 The difference of level of two points, A and B, is found by two routes whose lengths are D_1, D_2 kil respectively   If the discrepancy in the results is d, the m s e of one kilometre is $\dfrac{d}{\sqrt{D_1 + D_2}}$,

173 Adjustment of a Net of Levels.

—If a line of levels is to be run between two points a sufficient check of the accuracy of the work will in general be afforded by running the line over at least twice   Comparisons may be made at intermediate points not too far apart, and if the discrepancies found are within the limits already mentioned the mean of the results may be taken as giving the elevations sought   This method was used by the U S Engineers in determining the heights of the great lakes between Canada and the United States above mean tide

But when a complete topographical survey of a country is made, and a network of levels is necessary, an additional control of the accuracy of the work is afforded by the polygonal closing of the level lines forming the net; that is, from the condition that on passing round the polygon and

arriving at the starting-point we should have the same
height as at first   This assumes that the points of the net
are in the same level surface and that error of closure de-
pends upon errors of observation only   In the usual case,
where the net is small and the country comparatively level,
such an assumption is quite allowable   On the other hand,
if the net is very large or the country mountainous, system-
atic sources of error, arising principally from the spheroidal
form of the earth and the deviation of the plumb-line, would
be introduced   The corrections resulting from these causes
would have to be computed and applied   A full investiga-
tion of this point will be found in *Astron Nachr*, Vols 80-84

The adjustment of a net of levels may be carried out in
a similar way to the adjustment of a triangulation   Thus
suppose that the lines of levels form a closed figure   The
conditions to be satisfied among the observed differences
of height may be divided into two classes

(a) Those arising from non-agreement of repeated meas-
urements of differences of height between successive bench
marks   The equations expressing these conditions corre-
spond to the local equations in a triangulation.

(b) Those arising from the consideration that on starting
from any bench mark and returning to it through a series
of bench marks, thus forming a closed figure, we should
find the original height   The resulting equations corre-
spond to the angle and side equations in a triangulation

If the circuit, instead of being a simple one, has tie lines,
the number of closure conditions is easily estimated   For
if s be the number of bench marks, and l the number of
lines levelled over, the number of lines necessary to fix the
bench marks is $s - 1$, and therefore the number of super-
fluous lines—that is, the number of closure conditions to be
satisfied among the differences of height—is

$$l - s + 1$$

174 *Approximate Methods of Adjustment* —The differences
of height between successive bench marks may be found

by taking the means of the observed values, as explained in Art 172   Taking these means as independently observed quantities, they may be adjusted for non-satisfaction of closure conditions in various ways

(a) The differences of level between the successive bench marks will give rise to l observation equations   Taking the starting point as origin, $s - 1$ differences are required to fix the s stations   If these $s - 1$ differences are considered independent unknowns the remaining $l - s + 1$ unknowns may be expressed in terms of them, and the solution completed by the method of Art 109

(b) Since each independent polygon in the net contains one superfluous measurement, it gives a condition equation   With s stations connected by l lines there must be $l - s + 1$ condition equations, as the number of superfluous measurements is $l - s + 1$ The solution may be completed by the method of correlates, Art 110

As to the relative advantages of the two forms of solution, if the adjusted values only are to be found, without their weights, the test will be furnished by the number of normal equations in each case, as the principal part of the labor consists in solving the normal equations   The number by the first form is $s - 1$, and by the second $l - s + 1$, and therefore, in an extensive net with few tie lines, the method of condition equations is to be preferred, and *vice versa*

(c) Adjust each simple circuit in order by the principle of Art 115, and repeat the process until the required accuracy is reached

(d) The method employed on the British Ordnance Survey in reducing the principal triangulation   (See Art 152)   This method is easily applied and gives results practically close enough with the first approximation   It deserves to be employed much more than it is in work of this kind.

The Ordnance Survey levels, however, were not reduced by this method   " The discrepancies in the levelling along the different lines or routes brought out in closing on

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common points have been treated for England and Wales as a whole and rigorously worked out by the method of least squares, involving ultimately the solution of a system of equations with 91 unknown quantities " [x]

Ex In the figure $AWMG$, A is the initial point, height zero, and the measured heights and distances are as follows

Fig 51
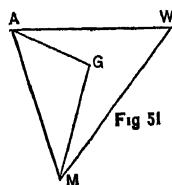
$AW = 42\ 65101$		$D_1 = 37\ 8$ kil
$AM = 54\ 74663$		$D_2 = 35\ 8$ "
$AG = 58\ 56223$		$D_3 = 22\ 6$ "
$WM = 12\ 10530$		$D_4 = 44\ 2$ "
$MG = 3\ 79892$		$D_5 = 27\ 9$ "

required the adjusted values of the heights

First Solution —Let the most probable corrections to the five measured differences of height in order be v_1, v_2, v_3, v_4, v_5 The points W, M, G are completely determined by AW, AM, AG We may, therefore, take v_1, v_2, v_3 as independent unknowns

The observation equations are

$$v_1 = v_1 \qquad \text{weight } 26\ 5$$
$$v_2 = v_2 \qquad " \quad 27\ 9$$
$$v_3 = v_3 \qquad " \quad 44\ 2$$
$$v_4 = -v_1 + v_2 - 0\ 00968 \qquad " \quad 22\ 6$$
$$v_5 = -v_2 + v_3 + 0\ 01668 \qquad " \quad 35\ 8$$

the weights being computed from $\dfrac{1000}{D}$

The solution is finished as in first solution, Art 140

Second Solution —From the closed circuits AWM, AMG we have the condition equations

$$v_1 - v_2 + v_4 = -0\ 00968$$
$$v_2 - v_3 + v_5 = +0\ 01668$$

with

$$\left[\frac{v^2}{D}\right] = a\ \min$$

The solution is finished as in second solution, Art 140

* *Abstract of Levelling in England and Wales* Introduction, p vi

Thi) d Solution —Adjust each simple closed circuit in the figure in order
Since in the circuit AMG the algebraic sum of the three differences of height
should be zero, we may apply the principles of Arts 110, 111   We have

$$AM = \quad 54\ 74663 \qquad D_2 = 35\ 8 \qquad p_2 = 27\ 9$$
$$MG = \quad 3\ 79892 \qquad D_5 = 27\ 9 \qquad p_5 = 35\ 8$$
$$AG = -\ 58\ 56223 \qquad D_3 = 22\ 6 \qquad p_3 = 44\ 2$$
$$\overline{-\ 0\ 01668} \qquad\qquad \overline{86\ 3}$$

Correction to $AM = \dfrac{35\ 8}{86\ 3} \times 0\ 01668 = +\ 0\ 00692$

Weight of adjusted $AM = 27\ 9 + \dfrac{1}{\dfrac{1}{35\ 8} + \dfrac{1}{44\ 2}}$

$$= 47\ 7$$

Hence for the circuit AMW

$$AM = 54\ 75355 \qquad p_2' = 47\ 7$$
$$AW = 42\ 65101 \qquad p_1 = 26\ 5$$
$$WM = 12\ 10530 \qquad p_4 = 22\ 6$$

which may be adjusted as above
The circuit AMG may be again adjusted, and so on

Fourth Solution

			I		
Weight	A	W	M	G	Means
26 5	0	42 65 01			
27 9	0		54 71663		
41 2	0			58 56223	
22 6		0	12 10530		
35 8			0	3 79892	
			II		
	0	12 65101			
	0		54 74663	58 56223	
	0	42 65101	54 75641		
			51 75096	58 54988	
Means	0	42 65101	51 75096	58 55670	
			III		
	0	0			0
	0		+ 0 00433	− 0 00553	+ 0 00216
	0				− 0 00276
		0	− 0 00535		− 0 00268
			0	+ 0 00682	+ 0 00341
			IV		
	0	12 65101			
	+ 0 00216		54 74879	58 55947	
	− 0 00276	42 64833	54 75363		
			54 75137	58 55329	
Means	− 0 00063	42 64978	51 75237	58 55670	
I nal values	0	12 65041	54 75400	58 55743	

Trigonometrical Levelling

175 In extensive surveys in which a primary triangulation is carried on, the heights of the stations occupied may be conveniently found by trigonometrical levelling   The extra labor required to measure the necessary vertical angles while the horizontal angles are being read at the several stations is but slight

When a country is hilly, heights can be perhaps as closely determined by trigonometrical levelling as by spirit levelling, but if the country is flat and the triangulation stations low, experience has shown that much more dependence is to be placed on the latter method   See, for comparisons of the relative accuracy of the two methods, Report of *U S Coast Survey*, 1876, App 16 and 17 ; Report of *G T Survey of India*, vol ii , Report of *New York State Survey*, 1882   Sometimes, indeed, the agreement is so close that it must be regarded as accidental   Thus in the determination of the axis of the St Gothard Tunnel Engineer Koppe found the difference of height of the two ends by trigonometrical levelling to be $39^m 13$ The spirit levelling of precision executed by Hirsch and Plantamour gave it $39^m 05$

The great cause of inaccurate work in trigonometrical levelling is atmospheric refraction   This is one of those disturbing causes which is so erratic in its character that no method has yet been devised for determining it that is very satisfactory   Hence the plan adopted in trigonometrical levelling is to observe only at or near the time of minimum refraction   Without this precaution very discrepant results may be looked for   For example, in India, where in certain districts the triangulation has been carried for hundreds of miles over a level country with stations 10 or 12 miles apart and from 18 to 24 feet high—just high enough to be mutually visible at the time of minimum refraction—"numerous instances are recorded of the vertical angles varying through a range of 6 to 9 minutes, corre

sponding to an apparent change in altitude of 100 to 150 feet in the course of 24 hours " [*]

176 *To Find the Zenith Distance of a Signal* —Point the telescope at a mark on the signal and bisect it with the horizontal thread Then turn the telescope 180° on its axis, transit it, and bisect the mark again At both bisections read the vertical circle and the level parallel to the vernier arm One half of the difference of the circle readings corrected for level will give a single determination of the value of the zenith angle sought The result is free from index error of the circle

Errors of graduation with good instruments are so small in comparison with the uncertainties arising from refraction that it is unnecessary to eliminate them in the measure of vertical angles

177 Let us consider the general case in which the zenith angles measured at two stations whose difference of height is required are not read simultaneously In the figure, if A_1, A_2 denote the positions of the instruments employed on two lofty stations, the angles that the observer has attempted to measure are $Z_1A_1A_2$ and $Z_2A_2A_1$ But on account of the refractive power of the atmosphere the path of a ray of light from A_1 to A_2 will not be a straight line but a curve more or less irregular, and the direction in which A_2 is seen from A_1 will be that of the tangent A_1T to this curve at A_1 The line of sight from A_2 to A_1 will not necessarily be over the same curve
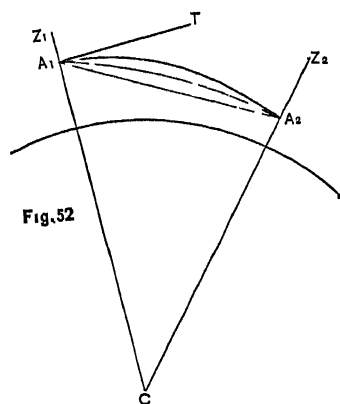
The angles between the apparent directions of the rays of light at A_1, A_2 and the real directions are called *refraction angles* Thus TA_1A_2 is the refraction angle at A_1

Let z_1, z_2 denote the measured zenith angles reduced to the heights of the instruments at A_1, A_2

$D =$ the distance between the instruments

$C =$ the angle at the earth's centre subtended by D

$\zeta_1, \zeta_2 =$ the refraction angles at A_1, A_2

Now, assuming the paths of the rays of light from one station to the other to be arcs of circles, which is approximately the case when the lines are of moderate length, we have

$$\zeta = \tfrac{1}{2}\frac{\text{path of ray}}{R_c}$$

where R_c is the radius of the refraction curve

But approximately

$$C = \frac{\text{path of ray}}{R}$$

where R is the mean radius of the earth

Hence we may put

$$\zeta_1 = \tfrac{1}{2}k_1 C \qquad \zeta_2 = \tfrac{1}{2}k_2 C \qquad (1)$$

where k_1, k_2 are constants and may be called *refraction factors*

Observations are frequently made so as to be simultaneous at A_1, A_2, and the line of sight may then be assumed to be the same arc of a circle for both directions. In this case if we put

$$\zeta_1 + \zeta_2 = kC$$

k is called the *coefficient of refraction*

178 **To Find the Refraction Factors.**- The triangle CA_1A_2 gives the relation

$$180° - (z_1 + \zeta_1) + 180° - (z_2 + \zeta_2) + C = 180° \qquad (1)$$

But, with sufficient accuracy, $C = \dfrac{D}{R \sin 1''}$, where R is the radius of curvature of the line in question

Hence, attending to equations 1, Art 177, we may write the above relation in the form

$$k_1 + k_2 = 2\left\{ 1 - \frac{R \sin 1''}{D}(z_1 + z_2 - 180°) \right\} \qquad (2)$$

This equation shows that a single line will not give the refraction factors, and we must, therefore, have a net of lines with the zenith angles read at the ends of each line If, for simplicity, we consider a quadrilateral $ABCD$ we shall have six equations of the form (2)   As these equations contain twelve unknowns we may, in order to reduce this number, assume, if the observations at station A over the lines AB, AC, AD are nearly simultaneous, that the k for each of these lines—that is, the k for station A —is the same   Similarly at stations B, C, D, so that in the most favorable case we shall have 4 unknowns and 6 equations

Fig 53
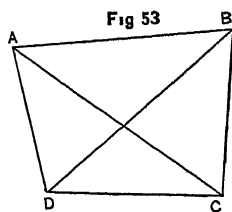
It may not always be possible to get one k for each station, but in a net, if all of the lines have been sighted over, a sufficient excess in number of equations over unknowns will in general be found to admit of solution by the method of least squares   As regards the weights to be assigned to these equations, we may proceed in the ordinary way   If z_1 were observed n_1 times, and z_2 were observed n_2 times, the weights of z_1 and z_2 may be taken to be n_1 and n_2 respectively, and the first equation would have a weight P given by

$$\frac{1}{P} = \frac{4R^2 \sin^2 1''}{D^2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

that is, its weight would be proportional to

$$\frac{n_1 n_2}{n_1 + n_2} D^2$$

179 To Find the Mean Coefficient of Refraction.
—When the zenith distances are simultaneous,

$$k_1 = k_2 = k \text{ suppose}$$

and the refraction coefficient at the moment of observation is for both stations,

$$k = 1 - \frac{R \sin 1''}{D}(z_1 + z_2 - 180°)$$

A value of k can thus be found for each line sighted over from both ends simultaneously   To get an average value of k for the whole system the weighted mean of these separate values must be taken   The same method of assigning weights may be used as in the preceding   Bessel argues that errors arising from irregularities in k are of much more importance than the errors in the zenith angles, and proposes the empirical formula *

$$\frac{n_1 n_2}{n_1 + n_2} \sqrt{D}$$

for assigning relative weights   In this he is followed by the Coast Survey in their determination of the coefficient of refraction from observations made in Northern Georgia near Atlanta base

The mean values of k found by the Coast Survey in N Georgia (1873) and by the New York State Survey (1882) were 0 143 and 0 146 respectively

180 To Find the Differences of Height —There are three cases to be considered

(1) When the zenith angles at both ends of each line are used

(a) When the zenith angles are not simultaneous

Let $H_1 =$ known height of first station
$H_2 =$ height of next station

* *Gradmessung in Ostpreussen*, p 196

From the triangle CA_1A_2 (Fig 52)

$$\frac{H_2 - H_1}{H_2 + H_1 + 2R} = \frac{CA_2 - CA_1}{CA_2 + CA_1} = \frac{\tan \frac{1}{2}(A_1 - A_2)}{\tan \frac{1}{2}(A_1 + A_2)} \quad (1)$$

Substituting for A_1, A_2 their values in terms of z_1, z_2, k_1, k_2, and reducing,

$$H_2 - H_1$$
$$= \left\{ D \tan \tfrac{1}{2}(z_2 - z_1) + \frac{k_2 - k_1}{4R}D^2 \right\} \left\{ 1 + \frac{H_2 + H_1}{2R} + \frac{D^2}{12R^2} \right\} \quad (2)$$

(b) When the zenith angles are simultaneous $k_1 = k_2$, and

$$H_2 - H_1 = D \tan \tfrac{1}{2}(z_2 - z_1)\left(1 + \frac{H_2 + H_1}{2R} + \frac{D^2}{12R^2} \right) \quad (3)$$

which is the form used on the Coast Survey

(2) When the angles observed at each end of a line are used separately

In the common case of a line sighted over from one end only, we have, since

$$A_1 = 180° - z_1 - \zeta_1$$
$$A_2 = \qquad z_1 + \zeta_1 - C$$

by substituting these values in (1) and reducing,

$$H_2 - H_1$$
$$= D \cot \left\{ z_1 - (1 - k)\frac{D}{2R \sin 1''} \right\} \left(1 + \frac{H_2 + H_1}{2R} + \frac{D^2}{12R^2} \right)$$

$$= D \cot z_1 + \frac{1 - k}{2R} D^2 + \frac{2 - k}{2R} D^2 \cot^2 z_1 \quad (4)$$

which is equivalent to the Coast Survey form

When the line is sighted over from both ends we have two values of the difference of height, whose weighted mean gives the required result

50

181 **Precision of the Formulas for Differences of Height.**—If h_1, h_2, h_3 denote the differences of height between two stations found from non-simultaneous readings of angles at the stations, from simultaneous readings, and from angles read at one of the stations only, respectively, then, with sufficient accuracy,

$$h_1 = D \tan \tfrac{1}{2}(z_2 - z_1) + (k_2 - k_1)\frac{D^2}{4R}$$

$$h_2 = D \tan \tfrac{1}{2}(z_2 - z_1) \tag{1}$$

$$h_3 = D \cot z_1 + (1 - k_1)\frac{D^2}{2R}$$

Let the m s e of an observed zenith angle be μ_z. Denote by μ_{h_1}, μ_{h_2}, μ_{h_3} the m s e of the differences of height found, and by μ_k the m s e of a refraction factor k. Then by differentiation, taking D, z_1, z_2, k_1, k_2 as independent variables and remembering that z_1, z_2 are each equal to $90°$ nearly, and that we may put $dD = 0$, since the distances are well known in comparison with the heights, we shall have (Art 65)

$$\mu_{h_1}{}^2 = \tfrac{1}{2} D^2 \sin^2 1'' \, \mu_z{}^2 + \tfrac{1}{8}\frac{D^4}{R^2} \mu_k{}^2$$

$$\mu_{h_2}{}^2 = \tfrac{1}{2} D^2 \sin^2 1'' \, \mu_z{}^2 \tag{2}$$

$$\mu_{h_3}{}^2 = D^2 \sin^2 1'' \, \mu_z{}^2 + \tfrac{1}{4}\frac{D^4}{R^2} \mu_k{}^2$$

These results show that differences of height are found with the greatest precision from simultaneous observations.

182 **Adjustment of a Net of Trigonometric Levels.**—It will be sufficient to consider a simple closed figure, as a net of levels can be broken up into a number of closed figures, generally triangles For simplicity take a triangle

The same value of k is assumed for all of the lines radiating from a station   Denote the values of k at the three stations by k_1, k_2, k_3 respectively   Then we have the observation equations

$$k_1 + k_2 = 2 \left\{ 1 - \frac{R \sin 1''}{D_1}(z_1 + z_2 - 180°) \right\} = l_1, \text{ a known quan.}$$

$$k_2 + k_3 = 2 \left\{ 1 - \frac{R \sin 1''}{D_2}(z_2 + z_3 - 180°) \right\} = l_2, \quad \text{``} \qquad \text{``}$$

$$k_3 + k_1 = 2 \left\{ 1 - \frac{R \sin 1''}{D_1}(z_3 + z_1 - 180°) \right\} = l_3, \quad \text{``} \qquad \text{``}$$

The condition to be satisfied among the differences of height at the vertices of the triangle is that, on starting from any station and returning to it through the other two stations, we should find the original height   Thus proceeding round the triangle in order of azimuth, if h_1, h_2, h_3 denote the differences of height of the stations, and (h_1), (h_2), (h_3) the most probable corrections to these values, we must have

$$h_1 + (h_1) + h_2 + (h_2) + h_3 + (h_3) = 0$$

that is, we must have a condition equation of the form

$$ak_1 + bk_2 + ck_3 = l_4$$

where a, b, c, l_4 are constants

The four equations may be solved by the method of correlates, and the differences of height may next be computed from equations 1, Art 181, and will be found consistent

If the circuit, instead of being a simple one, has diagonals, then, as in Art 173, if l is the number of lines read over, and s the number of stations in the circuit, the number of conditions to be satisfied among the differences of height is

$$l - s + 1$$

Ex In the triangulation of Georgia, near Atlanta Base, there were measured zenith angles as follows *

Stations occupied	Station read to	Zenith Angle	Time
S W Base,	Sweat Mt	89° 37' 53" 7	2 days
	Stone Mt	89° 26' 21" 6	
N E Base,	Sweat Mt	89° 39' 51" 7	1 "
	Stone Mt	89° 24' 04" 0	
Stone Mt ,	N E Base	90° 43' 20" 8	2 "
	S W Base	90° 41' 56" 8	
	Sweat Mt	90° 08' 58" 9	
Sweat Mt ,	S W Base	90° 33' 37" 2	1 "
	N E Base	90° 31' 43" 4	
	Stone Mt	90° 09' 37" 0	

The mean latitude is 34° N approx
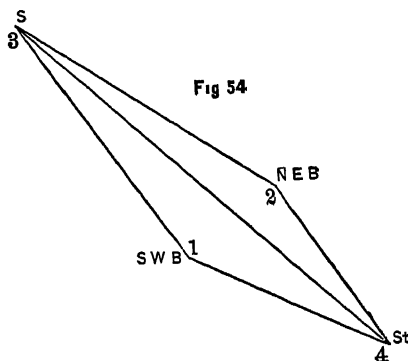
Fig 54

Distances

	m
S W Base–Sweat Mt ,	24368
N E Base–Sweat Mt ,	25203
N E Base–Stone Mt ,	16321
Stone Mt –S W Base,	17761
Sweat Mt –Stone Mt ,	40726

Call k_1, k_2, k_3, k_4 the refraction factors for all lines at the stations 1, 2, 3, 4 respectively

* The measures are taken from *C S Report*, 1876

We must form two classes of equations

(1) Observation equations   These are computed from the form

$$k + k' = 2\left\{1 - \frac{1}{BD}(z + z' - 180^\circ)\right\}$$

where $B = \dfrac{1}{R \sin 1''}$

The weights are computed according to the formula $\dfrac{n_1 n_2}{n_1 + n_2} \sqrt{D}$ (Art 179)
It will be close enough to take n_1, n_2 as the number of days of observation
That is the best we can do with our data   We have then the observation
equations

$$
\begin{array}{lll}
k_1 + k_3 = 0\ 249 & \text{weight} & 1\ 04 \\
k_1 + k_4 = 0\ 267 & \text{``} & 1\ 33 \\
k_3 + k_4 = 0\ 308 & \text{``} & 1\ 35 \\
k_2 + k_3 = 0\ 297 & \text{``} & 0\ 80 \\
k_2 + k_4 = 0\ 317 & \text{``} & 0\ 85 \\
\end{array}
$$

(2) Condition equations

$$
\begin{aligned}
\text{The number of lines} &= 5 \\
\text{The number of stations} &= 4 \\
\text{The number of condition equations} &= 5 - 4 + 1 \\
&= 2
\end{aligned}
$$

These two condition equations arise from the sums of two sets of three equations, each of the form

$$H - H' = D \tan \tfrac{1}{2}(z - z') + \frac{D^2}{4R}(k - k')$$

They are, for the triangles 134, 234,

$$
\begin{aligned}
0 &= -1\ 52 - 10\ 92\,k_1 - 41\ 79\,k_3 + 52\ 71\,k_4 \\
0 &= +1\ 84 + 14\ 47\,k_2 + 40\ 17\,k_3 - 54\ 64\,k_4 \\
\end{aligned}
$$

The solution of these equations, subject to the relation

$$[kk] = \text{a minimum,}$$

gives

$$
\begin{array}{ll}
k_1 = 0\ 118 & k_3 = 0\ 149 \\
k_2 = 0\ 107 & k_4 = 0\ 171 \\
\end{array}
$$

whence the adjusted differences of height follow at once

$$
\begin{array}{ll}
\overset{m}{} & \\
1 \text{ to } 3 & +198\ 24 \\
3 \text{ to } 4 & -\ \ \ 2\ 30 \ \Big\} \ \text{check sum}\quad 0\ 00 \\
4 \text{ to } 1 & -195\ 94 \\
3 \text{ to } 2 & -191\ 17 \\
2 \text{ to } 4 & +188\ 88 \ \Big\} \ \text{check sum} + 0\ 01 \\
4 \text{ to } 3 & +\ \ \ 2\ 30 \\
\end{array}
$$

The precision of the adjusted values, or of any function of them, may be
found exactly as in Art 114

183 Approximate Method of Adjusting a Net.—
On account of the many uncertainties attendant on finding
the refraction factors, it is not often that so elaborate a
method of adjusting the heights as the preceding is fol-
lowed

In the ordinary method of observation, where the ob-
served zenith angles are simultaneous at every two stations,

$$h = H_2 - H_1 = D \tan \tfrac{1}{2} (z_2 - z_1)$$

and the differences of height may be computed at once with-
out any reference to the coefficient of refraction   These
differences of height, considered as observed quantities,
may be adjusted for conditions of closure in the net, as in
spirit levelling, Art 174

The weights P to be assigned to the differences of height
in the solution will be found from

$$\frac{1}{P} = \mu_h^2 = \tfrac{1}{2} D^2 \sin^2 1'' \mu_z^2$$

and therefore the weights are inversely proportional to the
squares of the distances between the stations

When the zenith angles are not simultaneous, after find-
ing the refraction factors, as in Art 182, and computing
the differences of height, we should find the weights of
these differences of height from

$$\frac{1}{P} = \tfrac{1}{2} D^2 \sin^2 1'' \mu_z^2 + \tfrac{1}{8} \frac{D^4}{R^2} \mu_k^2$$

It would seem safe to assume $\mu_z = 2''$, $\mu_k = 0\ 02$

Now, $\tfrac{1}{2} D^2 \sin^2 1'' \mu_z^2 \gtrless \tfrac{1}{8} \frac{D^4}{R^2} \mu_k^2$

as $D \lessgtr 4$ miles

Hence for distances between the stations up to 4 miles the
first term is the important one, and for greater distances

the second term   We should, therefore, for distances between stations not greater than 4 miles, weight inversely as the square of the distance, and for distances over that amount inversely as the fourth powers of the distances

Ex 1 In the determination of the axis of the St Gothard Tunnel (Fig 37) the heights of the trigonometrical stations were determined by trigonometrical levelling   The following were the results unadjusted with their weights   Required the values adjusted for closure of circuits

		Diff of height	Wt			Diff of height	Wt
Airolo–	XII	914 96	23	III –	IX	216 46	1
Airolo–	X	1287 75	17	V –	IX	899 87	1
Airolo–	XI	1299 27	2	V –VIII		1075 77	1
Airolo–	IX	1553 09	5	III –VIII		391 74	1
XII –	X	372 73	5	VII –VIII		901 78	1
XII –	XI	384 41	2	V –	VII	174 45	7
XII –	IX	638 30	3	IV –	III	296 69	60
XII –VIII		814 35	1	VII –	III	509 49	4
X –	XI	11 60	3	Goschenen–	III	1376 19	14
X –	IX	265 48	6	V –	IV	387 24	20
X –VIII		441 10	2	VII –	IV	212 75	7
XI –	IX	253 87	1	Goschenen–	IV	1079 50	30
XI –VIII		429 55	10	Goschenen–	V	692 35	15
IX –VIII		175 37	1				

The heights of Airolo and Goschenen are $1147\,12$ and $1108\,07$ respectively, as found by spirit levelling, and are unaltered in the adjustment   The adjusted heights of the stations will be found to be

III	2484 26	IX	2700 35
IV	2187 57	X	2434 87
V	1800 36	XI	2446 49
VII	1974 78	XII	2062 08
VIII	2876 07		

Ex 2 From

$$H_2 - H_1 = D \tan \tfrac{1}{2}(z_2 - z_1)\left(1 + \frac{H_2 + H_1}{2R} + \frac{D^2}{12R^2}\right)$$

deduce

$$H_2 - H_1 = D \tan \tfrac{1}{2}(z_2 - z_1) + \frac{H_2^2 - H_1^2}{2R} + \frac{D^2}{12R^2}(H_2 - H_1)$$

Ex 3 If a series of points connected by observed zenith angles begin and end with points whose heights are known then the number of conditions to be satisfied among the differences of height is

$$l - s + 2$$

where l is the number of lines read over and s the number of points in the net

Ex 4 If zenith angles are read to two stations from a station between, show that the difference of height h, of the two stations will be found from

$$h = D_2 \cot z_2 - D_1 \cot z_1 - \frac{{}^{\bullet}D_2{}^2 - D_1{}^2}{2R} k$$

where D_1, D_2 are the distances from the station occupied

Also show that when $D_1 = D_2$ the precision of the height h is the same as if the observations had been made simultaneously at the two extreme stations themselves

CHAPTER IX

Observations for the determination of errors of gradua-
tion of line measures, and observations for the calibration
of thermometers, may be discussed together, as there is no
essential difference in the method of reducing them

184 **Line Measures.**—Let AB be a line measure
divided into n equal parts as nearly as may be at the points
1, 2, $n - 1$, and let o and n be the initial and terminal
marks   Comparisons are sup-
posed to have been made be-
tween AB and a standard of
length, so that the distance between o and n is known
The problem proposed is to find the corrections to the
intermediate graduation marks—that is, the amounts by
which the positions of the marks should be changed to be
in their true relative positions   When the corrections have
been applied to the distances o 1, 1 2,   these distances
should be all the same proportional part of the entire dis-
tance AB
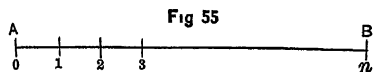
In making the necessary observations two methods are
in common use

(a) Two microscopes furnished with micrometers are
firmly mounted on a frame separate from the support on
which the graduated line measure rests   The zeros of the
micrometers are placed at a distance apart as nearly as
possible equal to the distances to be read   The marks
o, 1, 1, 2,   are in succession brought under the micro-
scopes and the micrometers read in each position   Each
distance is thus compared with the constant distance be-

tween the micrometer zeros, which differs from the distance of the true positions of the marks by a fixed but unknown amount

Let x_0, x_1, . x_n denote the corrections to the graduation marks at 0, 1, n Then for the first interval 0 1, if M_0, M_1 be the readings at 0, 1 and c be the unknown constant distance between the micrometer zeros, and d the distance between the corrected positions of the graduation marks, we should have

$$c + (M_1 + x_1) - (M_0 + x_0) = d$$

This may be written

$$x_0 - x_1 - y = l_{0\,1} \qquad\qquad (1)$$

where

$$l_{0\,1} = M_1 - M_0, \text{ and } y = c - d$$

Hence, taking four spaces only,

$$
\begin{aligned}
x_0 - x_1 &\qquad\qquad - y = l_{0\,1} \\
x_1 - x_2 &\qquad\qquad - y = l_{1\,1} \qquad\qquad (2)\\
x_2 - x_3 &\qquad - y = l_{2\,3} \\
x_3 - x_4 &- y = l_{3\,4}
\end{aligned}
$$

But as the distance between the initial and terminal points is known, the corrections x_0 and x_4 may each be taken equal to zero   Also, since the equations contain four unknowns and are themselves only four in number, we must either reduce the number of unknowns arbitrarily or make additional observations involving other combinations of the unknowns, in order to apply the method of least squares

It is usually more convenient to solve these equations by computing the corrections to the intervals 0 1, 0 2,   first of all, and then the corrections to the positions of the marks Writing z_1 for $x_1 - x_0$, z_2 for $x_2 - x_1$,   , and eliminating y,

we have from the above equations, supposing the first space compared with all of the others,

$$c_1 - c_2 = l_{1\,2} - l_{0\,1}$$
$$c_1 - c_3 = l_{2\,3} - l_{0\,1}$$
$$c_1 - c_4 = l_{3\,4} - l_{0\,1} \qquad (3)$$

But

$$c_1 + c_2 + c_3 + c_4 = x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + x_4 - x_3$$
$$= 0 \qquad (4)$$

Hence

$$c_1 = \tfrac{1}{4}\,(l_{1\,2} + l_{2\,3} + l_{3\,4} - 3l_{0\,1})$$
$$c_2 = \tfrac{1}{4}\,(l_{0\,1} + l_{2\,3} + l_{3\,4} - 3l_{1\,2}) \qquad (5)$$

Also $x_1 = c_1$, $x_2 = c_1 + c_2$,   , and are therefore known

Ex. In order to find the corrections to the double decimetre graduation marks needed to change the nominal into exact values in terms of the interval $0-1^m$ on a Repsold steel metre comparisons were made as follows   Two microscopes were mounted at a distance of 0^m 2 approx from each other, and readings made by pointing at the successive double decimetre marks, the microscopes remaining stationary while the metre was run under them   The order of reading was 0^m 0 and 0^m 2, 0^m 2 and 0^m 4, 0^m 8 and 1^m 0, 0^m 0 and 0^m 2, the interval 0^m 0 and 0^m 2 being read on at the beginning and end of each set of comparisons   Twenty-four sets were made and the following results obtained *

Interval,
	m		m		m		m		μ		μ
Interval,	0 2 to 0 4	=	0 0 to 0 2	+	2 1	$\pm$	0 1
"	0 4 to 0 6	=	0 0 to 0 2	+	2 6	$\pm$	0 1
"	0 6 to 0 8	=	0 0 to 0 2	+	0 7	$\pm$	0 1
"	0 8 to 1 0	=	0 0 to 0 2	+	2 2	$\pm$	0 1

With the above notation

$$z_1 - z_2 = \overset{\mu}{2}\ 1$$
$$z_1 - z_3 = 2\ 6$$
$$z_1 - z_4 = 0\ 7$$
$$z_1 - z_5 = 2\ 2$$
$$z_1 = +\ \overset{\mu}{1}\ 5,\ \text{and}\ x_1 = \overset{\mu}{1}\ 5$$
$$z_2 = -\ 0\ 6 \qquad x_2 = 0\ 9$$

* The value of the interval $0 - 1^m$ was found, by comparing with the German standard, to be $1^m + 86^\mu$ 18 $+ 10^\mu$ 65 t, where t is the temperature in degrees centigrade

or in tabular form,

Graduation	0^m 0	0 2	0 4	0 6	0 8	1^m 0
Correction	0^μ 0	$-1\,5$	$-0\,9$	$+0\,1$	$-0\,7$	0^μ 0

The Precision —Each of the intervals is entangled with the interval 0 0 and 0 2, and, therefore, the p e given are not independent   From the mode of measurement it is evident that the (p e)⁰ of 0 0 to 0 2 is half that of the (p e)² of each of the other intervals   Hence if r_1 is the p e of the first interval, and r the p e of each of the others,

$$r_1^2 = 2r^2$$
$$r_1{}^2 + r^2 = (0\ 1)^2$$
$$r^2 = 0\ 007 \qquad r_1^2 = 0\ 003$$

Hence
$$r_{x1}^2 = \tfrac{1}{25}(4 \times 0\ 007 + 16 \times 0\ 003)$$

and
$$r_{x1} = \pm 0^\mu\ 06$$

Similarly for the other marks

From equations 2 definite values of the corrections may be found if x_0 and x_4 are supposed known   There is, however, a greater probability of eliminating systematic error if, instead of spending all the time of observation in the direct comparison of the single intervals, we spend part of it in comparing combinations of those intervals   Let, then, as the best arrangement (see Art 153), the single intervals 0 1, 1 2, 2 3, 3 4, and all possible combinations of intervals, as 0 2, 1 3, 2 4, 0 3, 1 4, be equally well compared   The microscope intervals would, of course, be different in each set of comparisons, being approximately 0 1, 0 2, 0 3   The observation equations for this arrangement are

$$
\begin{aligned}
x_0 - x_1 && -y_1 && = l_{0\,1}\\
x_1 - x_2 && -y_1 && = l_{1\,2}\\
x_2 - x_3 && -y_1 && = l_{2\,3}\\
x_3 - x_1 - y_1 && && = l_{1\,4}\\
x_0 \quad\;\; -x_2 && -y_2 && = l_{0\,2}\\
x_1 \quad\;\; -x_3 && -y_2 && = l_{1\,3}\\
x_2 \quad\;\; -x_4 && -y_2 && = l_{2\,4}\\
x_0 \quad\;\; -x_3 && \;\; -y_3 && = l_{0\,3}\\
x_1 \quad\;\; -x_4 && \;\; -y_3 && = l_{1\,4}\\
x_0 \quad\;\; -x_1 && \;\; -y_1 = l_{0\,4}
\end{aligned}
\qquad (6)
$$

But x_0 and x_4 are known   Hence the normal equations

$$
\begin{aligned}
+4x_1 - x_2 - x_3 &\quad + y_2 + y_3 = -l_{01} + l_{12} + l_{13} + l_{14} \\
-x_1 + 4x_2 - x_3 &\quad\quad = -l_{02} - l_{12} + l_{23} + l_{24} \\
-x_1 - x_2 + 4x_3 &\quad - y_2 - y_3 = -l_{03} - l_{13} - l_{23} + l_{34} \\
&+4y_1 \quad\quad = -l_{01} - l_{12} - l_{23} - l_{34} \qquad (7)\\
+x_1 \quad\quad - x_3 &\quad + 3y_2 = -l_{02} - l_{13} - l_{21} \\
+x_1 \quad\quad - x_3 &\quad + 2y_3 = -l_{03} - l_{11}
\end{aligned}
$$

from which equations the most probable values of the corrections may be found

The Precision of the Corrections x_1, x_2, x_3 —The m s e of an observation of the unit of 'weight is found from the usual formula   We have

$$\mu = \sqrt{\frac{[vv]}{9-6}}$$

the number of observations being 9, and of independent unknowns 6

The weights of x_1, x_2, x_3 may be found by the methods of Chapter IV

The weight of x_1 and of x_3 is $\frac{20}{7}$, and the weight of x_2 is $\frac{20}{6}$ Hence the m s e of x_1, x_2, x_3 are known

Ex Solve by this method the example in Art 187

(b) The work of reduction is much facilitated by employing an auxiliary scale, *CD*, divided into spaces approximately equal to those of *AB*, and whose values have already been found by comparison with some standard   If, as before, we suppose the single spaces compared, and also all possible combinations of spaces, the observation equations would be the same as equations 6   The second scale, however, enables us to find each microscope interval employed   Hence y_1, y_2,   are known, and may, therefore, be transposed in these equations and added to the terms l_{01}, l_{12},   The observation equations thus involve only

x_0, x_4 as unknowns   Taking $x_0 = 0$, $x_1 = 0$, the normal equations are

$$+ 4x_1 - x_2 - x_3 = -l_{01} + l_{12} + l_{13} + l_{14}$$
$$- x_1 + 4x_2 - x_3 = -l_{02} - l_{12} + l_{23} + l_{24} \qquad (8)$$
$$- x_1 - x_2 + 4x_3 = -l_{03} - l_{13} - l_{23} + l_{34}$$

from which x_1, x_2, x_3 may be found

The reduction may, however, be made more easily by the application of the artifice employed in Art 156   If we consider x_0 and x_4 as yet unknown, the normal equations may be written

$$4x_0 - x_1 - x_2 - x_3 - x_4 = l_{01} + l_{02} + l_{03} + l_{04}$$
$$-x_0 + 4x_1 - x_2 - x_3 - x_4 = -l_{01} + l_{12} + l_{13} + l_{14}$$
$$-x_0 - x_1 + 4x_2 - x_3 - x_4 = -l_{02} - l_{12} + l_{23} + l_{24} \qquad (9)$$
$$-x_0 - x_1 - x_2 + 4x_3 - x_4 = -l_{03} - l_{13} - l_{23} + l_{34}$$
$$-x_0 - x_1 - x_2 - x_3 + 4x_4 = -l_{04} - l_{14} - l_{24} - l_{34}$$

Adding these equations, there results

$$0 = 0$$

This was to be expected, because we have left the initial and terminal points free   Hence we may assume any arbitrary relation between the corrections   Let us assume as most convenient

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0 \qquad (10)$$

and then by adding this relation to each of the normal equations we have

$$5x_0 = + l_{01} + l_{02} + l_{03} + l_{04}$$
$$5x_1 = - l_{01} + l_{12} + l_{13} + l_{14}$$
$$5x_2 = - l_{02} - l_{12} + l_{23} + l_{24} \qquad (11)$$
$$5x_3 = - l_{03} - l_{13} - l_{23} + l_{34}$$
$$5x_4 = - l_{04} - l_{14} - l_{24} - l_{34}$$

The whole solution may be conveniently arranged in tabular form   The sums of the horizontal rows are first found and then placed in the proper vertical columns, with

signs changed   The vertical columns are next added and divided by 5

	$-l_{0\,1}$	$-l_{0\,2}$	$-l_{0\,3}$	$-l_{0\,4}$	Sum$_1$
	$-l_{1\,2}$	$-l_{1\,3}$	$-l_{1\,4}$		Sum$_2$
		$-l_{2\,3}$	$-l_{2\,4}$		Sum$_3$
			$-l_{3\,1}$		Sum$_4$
$-Sum_1$	$-Sum_2$	$-Sum_3$	$-Sum_4$		
$5x_0$	$5x_1$	$5x_2$	$5x_3$	$5\lambda_4$	
x_0	x_1	x_2	λ_3	x_4	

By putting $x_0 = 0$, $x_4 = 0$ in equations 11, it is easily seen that we have the same results for x_1, x_2, x_3 as found from equations 8

Ex The intervals 0–5 mm , 5–10 mm ,   25–30 mm on a steel metre were compared with the interval 92–94 hundredths on a standard inch   The whole intervals 0–30 mm being known from other comparisons, show that the

p of the interval 0–5 mm , if the p c of a comparison is ϵ, is $\frac{\epsilon}{6} \sqrt{30}$

[Call metre intervals s_1, s_2, s_6, and inch interval a Then

$$s_1 = a + \varDelta_1$$
$$s_2 = a + \varDelta_2$$

$$s_6 = a + \varDelta_6$$

where $\varDelta_1$, $\varDelta_2$, $\varDelta_6$ are the errors
By addition,

$$[s] = 6a + [\varDelta]$$

$$a = \frac{[s]}{6} - \frac{[\varDelta]}{6}$$

where $[s]$ is a known quantity
Hence eliminate a from the value of s_1]

185 Calibration of Thermometers.—In Fig 55 let o, 1, 2, n denote the graduation marks cut on the glass stem, AB, of a thermometer   The point A we will take to be the freezing point and B the boiling point   These points are known, being first determined by the maker of the in-

strument, and can be redetermined at any time by special experiments

We will suppose that the errors of the freezing and boiling point marks are known, and proceed to consider the corrections to the intermediate marks due to want of uniformity of the bore of the stem, or the *calibration* corrections, as they are called   If the bore between two assigned marks were uniform it could be filled by a certain volume of mercury   The length of this standard volume or column would be indicated by the readings at the two marks   If the column were moved along until it came between two other marks with the same difference of readings as before, and the bore were not uniform with the bore at first position, the column would not fill the stem between the marks   The amount of difference would be the calibration correction for this interval

In explaining the method of determining the calibration corrections let us for simplicity consider 3 intermediate points only between the freezing and boiling points, that is, 4 intervals, 0 1, 1 2, 2 3, 3 4, of 45° each on Fahrenheit's scale   A column of mercury, of volume sufficient to fill 0 1 as nearly as may be, is broken off and the ends read when in the positions 0 1, 1 2, 2 3, 3 4   Another, equal to 0 2, is read in the positions 0 2, 1 3, 2 4, and a third, equal to 0 3, in the positions 0 3, 1 4 *

As it is impossible to break off the exact column required in every case, we break off one as nearly equal to it as possible and neglect the error introduced by the small discrepancy, which need not exceed 0° 2

186   The following method of breaking off column lengths was that employed by Mr Charles C Brown, of the Lake Survey, in determining the calibration corrections of thermometer 5280 Green (New York)

A column of mercury 200° to 250° in length was easily obtained by making that amount of mercury run into the

---

* This method of making the readings is known as Neumann's   See the corresponding form in triangulation Art 154

stem, a few slight jars being sufficient to start the column moving, a little manipulation then brought the empty space in the bulb to the junction of the bulb and the stem, when a sudden turn of the thermometer upright broke off the column, and almost as sudden an inversion preserved it When too large a space was left in the bulb to be filled by heating the thermometer to 140° or 150°, a few drops of mercury were allowed to drop off the end of the column in the stem held upright, good care being taken to stop the operation before the column joined the mercury in the bulb Then, the thermometer being heated until the mercury from the bulb began to appear in the stem, the column already in the stem was run down carefully, and partially joined to the mercury in the bulb, leaving a small bubble on one side of the column, the thermometer being allowed to cool slowly until the desired length of column above this bubble (which remains very nearly stationary) was obtained   The column was broken off at the bubble by a slight twitch or jar* If there are objections to heating the thermometer above a certain temperature, column lengths above 10° to 20° or 30° longer than the number of degrees of that temperature, depending on the distance of the 32° point from the bulb, can be obtained by jarring off small drops of mercury from a long column into the reservoir at the top of the stem   It requires much more time, care, and patience to obtain a column in this way than in the other   Columns more than about 160° in length were so obtained in this case   It is rather difficult to break off short columns 5° to 15° in length in the manner first described, the weight of mercury in the short column not giving momentum enough to move it away from the rest of the column readily   A little patience is all that is necessary, however   To be able to lead the shorter columns at 32° a column 10° to 50° long, depending on the temperature at the time, must be broken off and put into the reservoir at the upper end of the tube, out of the way

187   Let x_0, x_1, x_2, x_3, x_4 denote the calibration correc-

tions at the several graduation marks   Then, approxi-
mately, we have for each interval a relation of the form

$$\text{column} = \text{diff of readings} + \text{diff of cal corr}$$

As the volumes of the columns broken off correspond to the
constant interval between the microscopes in the case of
line measures, the observation equations furnished by the
thermometer readings will correspond in form to equations
6, Art 184

The rigorous solution of these equations is, however,
rather complicated, and though in comparisons of stand-
ards of length it may be allowable, from the precision with
which measurements can be made, to spend the labor de-
manded by this form of reduction, yet in thermometric work,
where the readings cannot be very close from the nature of
the case, an approximate form is sufficient   By the following
artifice, which is quite analogous to that introduced by
Hansen in the adjustment of a triangulation (Art 155), the
labor is reduced very materially

Instead of finding the corrections to the graduation
marks directly, the corrections to the several intervals be-
tween the graduation marks may first be found, and thence
the corrections to the marks at the ends of the intervals
Thus if z_1, z_2, z_3, z_4 denote the corrections to the 4 intervals
in our example, then

$$
\begin{aligned}
z_1 &= x_1 - x_0 & z_3 &= \lambda_3 - r_2 \\
z_2 &= x_2 - x_1 & z_4 &= 1_4 - 1_3
\end{aligned}
\tag{1}
$$

From the observation equations 6 we have, by subtract-
ing in pairs,

$$
\begin{aligned}
z_1 - z_2 &&&= l_{1\,0} - l_{0\,1} \\
&z_2 - z_3 && = l_{2\,3} - l_{1\,2} \\
&&z_3 - z_4 = l_{3\,1} - l_{1\,3} \\
z_1 && - z_3 &= l_{1\,3} - l_{0\,3} \\
&z_2 && - z_4 = l_{2\,4} - l_{1\,1} \\
z_1 && - z_4 &= l_{1\,4} - l_{0\,1}
\end{aligned}
\tag{2}
$$

Hence, considering $l_{12} - l_{01}$, $l_{21} - l_{12}$, as independently observed quantities, we have the normal equations

$$
\begin{aligned}
3z_1 - z_2 - z_3 - z_4 &= (l_{12} - l_{01}) + (l_{11} - l_{02}) + (l_{11} - l_{04}) \\
-z_1 + 3z_2 - z_3 - z_4 &= -(l_{12} - l_{01}) + (l_{22} - l_{12}) + (l_{21} - l_{11}) \\
-z_1 - z_2 + 3z_3 - z_4 &= -(l_{22} - l_{12}) + (l_{34} - l_{23}) - (l_{13} - l_{02}) \\
-z_1 - z_2 - z_3 + 3z_4 &= -(l_{34} - l_{23}) - (l_{21} - l_{12}) - (l_{14} - l_{03})
\end{aligned}
\tag{3}
$$

which equations when added give $0 = 0$ identically. The reason is that we have not yet fixed the initial or terminal points. To do this we may, as most convenient, assume the relation

$$z_1 + z_2 + z_3 + z_4 = 0 \tag{4}$$

Adding this relation to each of the normal equations, we find the values of the unknowns. Thus

$$
\begin{aligned}
4z_1 &= (l_{12} - l_{01}) + (l_{11} - l_{02}) + (l_{11} - l_{04}) \\
4z_2 &= -(l_{12} - l_{01}) + (l_{21} - l_{11}) + (l_{21} - l_{11}) \\
4z_3 &= -(l_{21} - l_{12}) + (l_{31} - l_{23}) - (l_{11} - l_{02}) \\
4z_4 &= -(l_{11} - l_{21}) - (l_{21} - l_{13}) - (l_{11} - l_{01})
\end{aligned}
$$

The computation of the unknowns may be much facilitated by arranging in tabular form, as follows

z_1	z_2	z_3	z_4	
	$-(l_{12} - l_{01})$	$-(l_{11} - l_{02})$	$-(l_{11} - l_{04})$	Sum₁
		$-(l_{21} - l_{12})$	$-(l_{11} - l_{11})$	Sum₂
			$-(l_{11} - l_{23})$	Sum₃
$-$ Sum₁	$-$ Sum₂	$-$ Sum₃		
$4z_1$	$4z_2$	$4z_3$	$4z_4$	
z_1	z_2	z_3		

The corrections x_1, x_2, x_3 are then found from the relations

$$
\begin{aligned}
x_1 &= z_1 \\
x_2 &= x_1 + z_2 \\
x_3 &= x_2 + z_3
\end{aligned}
$$

since the correction x_0 may be assumed to be zero, and then r_4 must also be zero from the relations (1) and (4)

As a check we notice that the values of c_4 and x_3 must be equal but of opposite sign

The Precision —If v_1', v_2',   denote the residuals of equations 2, we have for the m s e of a column length

$$\sqrt{\frac{[v'v']}{6-3}}$$

the number of equations being 6 and of independent unknowns 3

Now, since the column length involves the difference of the calibration corrections at its ends, if we assume the m s e of these corrections to be equal we shall have the m s e of a calibration correction by dividing the above result by $\sqrt{2}$, thus

$$\mu = \sqrt{\frac{[v'v']}{2(6-3)}}$$

The weights of the unknowns or of any functions of them may be found by the methods of Chapter IV

Ex The following were the observed values of the lengths of the 45, 90', and 135° columns of thermometer Green 4470, made to determine the calibration corrections at the 77°, 122°, and 167° points

45° col	90° col	135' col
44° 68	90° 07	134° 61
44° 71	90° 09	134° 68
44° 72	90° 11	
44° 75		

The corrections at the freezing and boiling points are known

CHAPTER X

189 In all discussions hitherto we have considered the observed quantity, whether a function of one or more variables, to be a function whose form was known and which could therefore be developed in terms of that variable   In the physical sciences we meet with a different problem   From observation we have values of a function corresponding to certain known values of a variable, and are required to determine the form of the function from the observed values, in other words, to find the algebraic formula connecting the function and the variable   The method of least squares will not enable us to find the most probable form of this function   All it will do is to show how to approach more closely to its form after that form has been found approximately by other means

For example, the observed height of the tide at a place may be considered a function of the time of day   If observations are made with a staff gauge three or four times a day, and the stage of water at a time not observed is required, we cannot interpolate between the observed values in the most probable manner till the function connecting the height and the time of day is known

Sometimes, indeed, the observations may be so taken as to record themselves   Thus with Saxton's self-registering tide gauge, in common use in the United States, the curve representing the rise and fall of the water is traced continuously on a web of paper moving uniformly past a pencil point by means of clock-work   Plotting the time along one edge of the paper as abscissa, the stage of water at

any time required can be read at once from the sheet
The curve representing the rise and fall of the water is
in this case completely known, and no computation is
necessary

190 From the nature of physical phenomena, in which
observations are made only at intervals, and into which sys-
tematic error largely enters, we must admit that it is most
probable that the function connecting the observed quantity
and the variable is different for each observation   We
should thus have a series of equations of the form

$$M_1 = f_1(x, a, b, c, \qquad)$$
$$M_2 = f_2(x, h, k, l, \qquad)$$

where M_1, M_2,   are the observed values, x the variable,
and a, b,   , h, k,   are constants

In order to apply the method of least squares to de-
termine the constants, the functions must first be reduced
to the linear form, and the number of quantities to be de-
termined must be made less in number, arbitrarily it may be,
than the number of observations   The first point is to
determine as nearly as we can the general form of the
functions connecting the observed quantities and the vari-
able

For this no general rule can be given   From theoretic
considerations we may hit upon a certain formula as
plausible   It must then be found by trial how closely it
fits the observed values   It is often very convenient to
make a graphical representation, the variable being the
abscissa, and the observed values of the function correspond-
ing to known values of the variable the ordinates   The
points plotted will in general fall so irregularly, here a
group and there a vacant space, that a free-hand curve only
can be drawn which will conform to the general outline of
the plot   The curve will represent the general form of the
function we are seeking   This method of eliminating ir-
regularity by drawing a mean curve to represent the value

of the function corresponds to assuming a common form for
the algebraic functions $f_1, f_2,$ above

A good example is afforded by the method employed by
Humphreys and Abbot in their Mississippi River work for
finding the law of velocities below the surface in a plane
parallel to the current * The observations for velocity
were made with floats at different depths, started from
boats anchored at various distances from the shore   In the
first place, all of the velocities observed from each anchorage
were plotted on cross-section paper, the depths of the floats
being the ordinates and the velocities the abscissas   The
resulting curves showed marked irregularities of form   To
eliminate errors of observation the velocities were grouped
in sets corresponding to nearly equal depths of water and
to nearly equal velocities of the river, and the means plotted
These curves indicated a law, though not sufficiently clearly
to allow of deducing an algebraic expression for it   " It is
manifest that some further combination is necessary in
order to eliminate the effect of disturbing causes   Since
the absolute depths differ, this can only be done by com-
bining the velocities at proportional depths "   A curve
was therefore, plotted from the mean of all the velocities at
proportional instead of at absolute depths   This curve
proved to be approximately a parabola with axis horizontal
and vertex about o 3 of the depth of the river below the
surface   It could, therefore, be expressed by an algebraic
formula

In many cases we may find the form of the function
over which the observations extend by means of a formula
of interpolation   Thus if x' is an approximate value of x,
then for observations in the vicinity of x we have, by
Taylor's theorem,

$$f(x) = f\{x' + (x - x')\}$$
$$= f(x') + b(x - x') + c(x - x')^2 +$$

where $b, c,$. are unknown constants to be determined

Having found the form of the function approximately by such means as the preceding, we can, by the method of least squares, determine the most probable values of the constants involved, so far as the observations on hand are concerned   For let the function be reduced to the linear form, and the most probable values of the constants and the precisions of these values may then be found by the ordinary rules for observation equations as laid down in Chapter IV

191   The question naturally arises as to the number of terms of a series, such as the above, that should be taken in any special case   Cut and try is our main guide   If the plot of the observations shows, for example, that the phenomenon can be very closely represented by a straight line, we should take the first two terms   Thus it has been usual to take the relation between the length, V, of a bar of metal and the temperature, t, to be of the form

$$V = M + b(t - t')$$

where M is the observed length at the temperature t', and b is the coefficient of expansion to be found from the observations   More refined methods of observation indicate, however, that this expression is not sufficient to express the relation between length and temperature   The plot of the observed values indicates a curve of higher dimensions than the first, so that we must take additional terms of the series to represent it, thus

$$V = M + b(t - t') + c(t - t')^2 + . \quad .$$

A second guide is afforded by the value obtained of the sum of the squares of the residuals of the observation equations   If the nature of the observations is such as to warrant the expectation of a value much smaller than that obtained, we may take additional terms of the series, and if this is still unsatisfactory we must seek another function   This test may also be used in comparing one form of function with another

53

In any case, even when all of the tests applied appear satisfactory, we cannot say that our final result is the most probable value of the function that can be found   We have made too many assumptions for that   In the first place, the form of the function assumed as a first approximation is too uncertain, and, again, we have arbitrarily assumed the number of constants to be determined, and determined this limited number so as to satisfy a special set of observations only   All that our final formula will enable us to do is to interpolate in the most probable manner within the range of the special set of observed values, but not to extrapolate If, however, the formula has been tested by many series of observations made under the most diverse conditions, and is found to satisfy them well, we can with confidence apply it in cases where no observations have been made   We may, in fact, consider that we have found the law connecting the function and the variable

192 It is evident, too, that different experimenters may derive formulas widely different to represent like phenomena, and that each formula may satisfy the special set of observations from which it was derived tolerably well Each experimenter may have chosen his constants differently, as well as a different form of function to begin with As an example of this we may cite the formulas proposed [x] to represent the variation of the elastic forces of vapor at different temperatures t

Young proposed

$$P = (a + bt)^m$$

a and b being constants to be found from observation
  Roche,

$$P = a\, a^{\frac{\tau}{1 + m x}}$$

where $\lambda = t + 20°$
  Biot and Regnault,

$$\log P = a + b a^t + c\beta^t$$

* Mousson, *Physik*, vol ii Zurich, 1880

Of these formulas the first involves two constants, the second three, and the third five The first curve, therefore, requires two observations to determine it, the second three, and the third five   If a group of observations were plotted with temperatures as abscissas and pressures as ordinates, then, since the last formula represents a curve passing through five of these points, it is evident that if the observations are good this curve would in all probability pass near all intermediate points and more closely than a curve fixed by only two or three points   It should, therefore, be chosen

Ex 1 The following are the results of the observations made for velocity of current of the Mississippi River by Humphreys and Abbot at Carrollton and Baton Rouge in 1851   Each is the mean of 222 observations and is given at proportional depths, the whole depth being represented by unity

Propor depth of float below surface	Obs velocity in feet per second
0 0	3 1950
0 1	3 2299
0 2	3 2532
0 3	3 2611
0 4	3 2516
0 5	3 2282
0 6	3 1807
0 7	3 1266
0 8	3 0594
0 9	2 9759

In Art 191 it has been shown that the curve of velocities is approximately parabolic in form   As a first approximation, therefore, we take three terms of the expansion from Taylor's theorem,

$$V = a + bD + cD^2$$

where V is the velocity, D the proportional depth, and a, b, c constants to be determined from the observations

Hence the observation equations

$$3\ 1950 = a$$
$$3\ 2299 = a + 0\ 1b + 0\ 01c$$
$$3\ 2532 = a + \quad 2b + \quad 04c$$
$$3\ 2611 = a + \quad 3b + \quad 09c$$
$$3\ 2516 = a + \quad 4b + \quad 16c$$
$$3\ 2282 = a + \quad 5b + \quad 25c$$
$$3\ 1807 = a + \quad 6b + \quad 36c$$
$$3\ 1266 = a + \quad 7b + \quad 49c$$
$$3\ 0594 = a + \quad 8b + \quad 64c$$
$$2\ 9759 = a + 0\ 9b + 0\ 81c$$

To solve these equations proceed as in Art 104   Take the mean and subtract from each, and we have

$$+ 0\ 0188 = - 0\ 45b - 0\ 285c$$
$$+ 0\ 0537 = - 0\ 35b - 0\ 275c$$
$$+ 0\ 0770 = - 0\ 25b - 0\ 245c$$
$$+ 0\ 0849 = - 0\ 15b - 0\ 195c$$
$$+ 0\ 0754 = - 0\ 05b - 0\ 125c$$
$$+ 0\ 0520 = + 0\ 05b - 0\ 035c$$
$$+ 0\ 0045 = + 0\ 15b + 0\ 075c$$
$$- 0\ 0496 = + 0\ 25b + 0\ 205c$$
$$- 0\ 1168 = + 0\ 35b + 0\ 355c$$
$$- 0\ 2003 = + 0\ 45b + 0\ 525c$$

Hence the normal equations

$$- 0\ 2031 = 0\ 8250b + 0\ 7425c$$
$$- 0\ 2232 = 0\ 7425b + 0\ 7410c$$

and $\qquad\qquad b = 0\ 4424 \qquad c = - 0\ 7652 \qquad a = 3\ 1952$

$$V = 3\ 1952 + 0\ 4424D - 0\ 7652D^2$$

The residuals v are found by substituting for a, b, c in the observation equations, and give

$$[vv] = 0\ 57$$

If we take four terms of the expansion, so that

$$V = a + bD + cD^2 + dD^4$$

and proceed similarly, we shall find

$$a = 3\ 1935, \qquad b = +\ 0\ 4735 \qquad c = -\ 0\ 8563, \qquad d = +\ 0\ 0675$$

so that

$$V = 3\ 1935 + 0\ 4735\ D - 0\ 8563\ D^2 + 0\ 0675\ D^3$$

and the sum of the squares of the residuals v is given by

$$[vv] = 0\ 45$$

The sum of the squares of the residual errors being less than in the former case, we conclude that the observations are better represented by the formula last obtained

It is to be borne in mind that in the application of the method of least squares only one set of measured quantities, if more than one occur in the problem, can be considered variable   The error is thrown into this set Thus in the problem just solved the depths are supposed to be correctly measured and the error alone to occur in the measured velocities   So in finding the expansion of a body it is usual to consider all of the error as occurring in the observed lengths, and to take the thermometer readings to be correct   The justice of these assumptions may, however, very fairly be questioned   A discussion of the general problem covering such cases will be found in Art   106

Ex 2   The tension, H, of saturated steam at temperature $t°$ C is found from the formula *

$$H = a\ 10^{-\frac{bt + ct^2 +}{1 + at}} \tag{1}$$

where $\alpha = 0\ 0036678$, and $a, b, c,$ are constants, $n + 1$ in number, to be determined from observed values of t and H We have also given that when

$$t = 100°\ C\ ,\quad H = 760^{mm}$$

To find the constants a, b, c,   we proceed as follows

Substituting the values of t and H in equation 1, and eliminating a, we have

$$H = 760 \cdot 10^{-\left(\frac{100}{1+100a} - \frac{t}{1+at}\right)b - \left(\frac{100^2}{1+100a} - \frac{t^2}{1+at}\right)c} -$$

$$= 760 \cdot 10^{-(bp_1 + cp_2 + \quad)} \text{ suppose}$$

or

$$\log 760 - \log H = bp_1 + cp_2 + \tag{2}$$

Next compute approximate values of b, c,   by selecting n of the observations and substituting the observed values of t and H in Eq 2   Let b', c', denote the values found from the solution of the resulting equations, and let H' denote the corresponding value of H, so that

$$H' = 760 \cdot 10^{-(b'p_1 + c'p_2 + \quad)} \tag{3}$$

Also, let x, y,   denote the corrections to be added to the approximate values b', c',   to find the most probable values, so that

$$b = b' + x$$
$$c = c' + y$$

then, from (2),

$$\log 760 - \log\left\{H' + (H - H')\right\} = (b' + x)p_1 + (c' + y)p_2 +$$

or expanding by Taylor's theorem and remembering that, from (3),

$$\log 760 - \log H' = b'p_1 + c'p_2 +$$

we have

$$H - H' = \frac{H'}{\text{mod}} \; (p_1 x + p_2 y + \quad)$$

the linear form required   The solution may be finished in the usual way, as given in Chapter IV

193 The preceding examples show that as soon as it has been decided what formula will apply to a special set of observations the main difficulty is in reducing this formula to the linear form   When this has been done so that the formula is in the form of an ordinary observation equation, the reduction by the method of least squares is straightforward.

In cases where the data are insufficient or contradictory it is useless to waste time on long computations, as the graphical plot will show at a glance all that the observations can show   With cross section paper plots can be made rapidly, and by transferring to tracing linen one plot can be placed over another, so that comparisons can be readily effected and a mean value struck

In the problem of the law of velocities in rivers, after having decided from the various plots that the curve of velocities is approximately parabolic in form, it is better to employ the method of least squares to determine the constants of the curve rather than to trust to the graphical method throughout, but only for the reason that the data available are very complete

194 *Periodic Phenomena* —A large class of physical phenomena is more or less periodic in character   The daily temperature throughout the year at a given place, the error of graduation of the limb of a theodolite, the magnetic declination with reference to the time, etc, are examples The phenomenon may not be strictly periodic in that like periods succeed each other in their proper order, or that even any one period is perfect throughout   If a plot of the observed values of the function corresponding to certain values of the variable involved be made, and it indicates the periodic character of the function, we may assume as the form of the function a number of terms of the series furnished by Fourier's theorem   Each observation will give an observation equation, and from the observation equations the values of the constants in the formula will be determined   As in the cases already discussed, the final formula is to be looked on as holding only within the limits of the observations, serving as a guide for the future study of the phenomenon in question, and only to be used with great caution outside the limits of the observed values

Suppose, then, that n observations give the values $M_1, M_2,$ M_n corresponding to the values $\varphi, \varphi + \theta,$ . . $\varphi + (n - 1)\theta$ of the variable φ uniformly distributed over

the period represented by 360°, that is, such that the interval θ is $\dfrac{360°}{n}$

Now, if $f(\varphi)$ is any arbitrary periodic function of a variable φ, we may, according to Fourier's theorem, write

$$f(\varphi) = X + h_1 \sin(a_1 + \varphi) + h_2 \sin(a_2 + 2\varphi) +$$

where $X, h_1, h_2, \quad . \; a_1, a_2, \quad$ are constants to be determined from the observed values of $f(\varphi)$ corresponding to assigned values of the variable φ Substitute M_1, M_2, M_3, M_n for $f(\varphi)$, and $\varphi, \varphi + \theta, \varphi + 2\theta, \quad \varphi + (n-1)\theta$ for φ, in the above equation, and we shall have n equations from which to determine a number of constants not exceeding $n - 1$

To lighten the numerical work let us assume the arithmetic mean of the observed values as an approximate value of X, and let x denote the correction to this value, so that

$$X = \frac{[M]}{n} + x$$

Then if, as usual, is placed

$$M_1 - \frac{[M]}{n} = l_1, \quad M_2 - \frac{[M]}{n} = l_2,$$

the n observation equations may be expressed by the general formula

$$
\begin{aligned}
x + h_1 \sin(a_1 + \varphi + m\theta) & \\
+ h_2 \sin(a_2 + 2\varphi + 2m\theta) + & \quad - l_{m+1} = v_{m+1}
\end{aligned}
$$

where m assumes all values from o to $n - 1$

Writing them at full length,

$$x + y_1 \cos o + z_1 \sin o + y_2 \cos 2\, o + z_2 \sin 2\, o + \quad . - l_1 = v_1$$
$$x + y_1 \cos \theta + z_1 \sin \theta + y_2 \cos 2\, \theta + z_2 \sin 2\, \theta + \quad - l_2 = v_2$$

$v_1, v_2,$ being, as usual, the errors of observation, and

$$y_1 = h_1 \sin (a_1 + \varphi), \qquad y_2 = h_2 \sin (a_2 + 2\varphi),$$
$$z_1 = h_1 \cos (a_1 + \varphi), \qquad z_2 = h_2 \cos (a_2 + 2\varphi),$$

The normal equations reduce to the simple forms (see Art 9)

$$n\,x \qquad = [l] \quad = 0$$

$$\frac{n}{2} y_1 \qquad\qquad = \Sigma\, l_{m+1} \cos m\theta$$

$$\frac{n}{2} z_1 \qquad\qquad = \Sigma\, l_{m+1} \sin m\theta$$

$$\frac{n}{2} y_2 \qquad = \Sigma\, l_{m+1} \cos 2m\theta$$

$$\frac{n}{2} z_2 = \Sigma\, l_{m+1} \sin 2m\theta$$

$$\cdot \qquad \cdot \quad \cdot \qquad\qquad \cdot$$

where m has all values from 0 to $n-1$

Hence $h_1, h_2, \qquad a_1, a_2, \qquad$ are known, and their values, being substituted in $f(\varphi)$, give the function required

If the initial value of $f(\varphi)$ corresponds to $\varphi = 0$, that is, if the observed values $M_1, M_2, M_3, \qquad M_n$ of $f(\varphi)$ correspond to 0, θ, 2θ, . . . $(n-1)\theta$, where $n\theta = 360°$, it is simpler to write the equation for $f(\varphi)$ first of all in the form

$$f(\varphi) = X + y_1 \cos \varphi + z_2 \cos 2\varphi + \ldots$$
$$+ z_1 \sin \varphi + z_2 \sin 2\varphi +$$

where

$$y_1 = h_1 \sin a_1 \qquad y_2 = h_2 \sin a_2,$$
$$z_1 = h_1 \cos a_1 \qquad z_2 = h_2 \cos a_2,$$

Then with the value $\dfrac{[M]}{n}$ as the approximate value of X, we

54

have the n observation equations and the normal equations of the same forms as before

Hence the function is known

195 Two cases of frequent occurrence are

(a) $\quad n = 10, \quad \theta = \tfrac{360}{10}° = 36°$

The normal equations may be written

$5y_1 = l_1 - l_6 + \{(l_2 - l_7) - (l_5 - l_{10})\}\cos 36° + \{(l_3 - l_8) - (l_4 - l_9)\}\cos 72°$

$5z_1 = \qquad \{(l_2 - l_7) + (l_5 - l_{10})\}\sin 36° + \{(l_3 - l_8) + (l_4 - l_9)\}\sin 72°$

$5y_2 = l_1 + l_6 + \{(l_2 + l_7) + (l_5 + l_{10})\}\cos 72° - \{(l_3 + l_8) + (l_4 + l_9)\}\cos 36°$

$5z_2 = \qquad \{(l_2 + l_7) - (l_5 + l_{10})\}\sin 72° + \{(l_3 + l_8) - (l_4 + l_9)\}\sin 36°$

$. \qquad\qquad\qquad\qquad . \qquad\qquad . \quad .$

It will be noticed that the difference of the subscripts of the l's in each parenthesis is always five, the same as the coefficient of the unknown

(b) $\quad n = 12, \quad \theta = 30°$

The normal equations

$6y_1 = (l_1 - l_7) + \{(l_3 - l_9) - (l_5 - l_{11})\}\sin 30° + \{(l_2 - l_8) - (l_6 - l_{12})\}\cos 30°$

$6z_1 = (l_4 - l_{10}) + \{(l_3 - l_9) + (l_5 - l_{11})\}\cos 30° + \{(l_2 - l_8) + (l_6 - l_{12})\}\sin 30°$

$6y_2 = (l_1 + l_7) - (l_4 + l_{10}) + \{(l_2 + l_8) + (l_6 + l_{12}) - (l_3 + l_9) - (l_5 + l_{11})\}\sin 30°$

$6z_2 = \qquad\qquad \{(l_2 + l_8) - (l_6 + l_{12}) + (l_3 + l_9) - (l_5 + l_{11})\}\cos 30°$

$. \qquad\qquad\qquad . \qquad\qquad . \quad . \quad .$

The difference of the subscripts of the l's in each parenthesis is six in this case

196 *The Precision* —The m. s e of the unit of weight is found from the usual formula

$$\mu = \sqrt{\frac{[vv]}{n - n_s}}$$

where n_s is the number of constants determined

Check of $[vv]$ Generally (Art 100)

$$[vv] = [ll] - \frac{[al]^2}{[aa]} - \frac{[bl\ 1]^2}{[bb\ 1]} - \frac{[cl\ 2]^2}{[cc\ 2]} -$$

Now substitute for $[al]$, $[bl\ 1]$, $[cl\ 2]$,   their values from the normal equations above, and remembering that these equations contain only one unknown each,

$$[vv] = [ll] - \frac{0}{n} - \frac{(\frac{1}{2}ny_1)^2}{\frac{1}{2}n} - \frac{(\frac{1}{2}nz_1)^2}{\frac{1}{2}n} -$$

$$= [ll] - \frac{n}{2}[hh]$$

Ex 1   The mean monthly heights of the water in Lake Michigan at Chicago below the mean level of the lake from 1860 to 1875, for the 12 months of the year 1868, were as in column M of the following table

	M	l		M	l
	ft	*ft*		*ft*	*ft*
Jan	1 17	+ 0 45	July	0 11	− 0 61
Feb	1 25	+ 0 53	Aug	0 43	− 0 29
March	0 59	− 0 13	Sept	0 68	− 0 04
April	0 68	− 0 04	Oct	0 94	+ 0 22
May	0 29	− 0 43	Nov	1 05	+ 0 33
June	0 17	− 0 55	Dec	1 32	+ 0 60
			Mean,	0 72	

Required a formula from which the mean daily height may be found

The period is one year, and if we assume that its 12 months correspond to 360° and that the months are of equal length, each interval (θ) would be 30°

The values of the coefficients y_1, z_1,   can be at once written down from (b)   They are

$$y_1 = + 0\ 517 \qquad y_2 = − 0\ 010$$
$$z_1 = − 0\ 194 \qquad z_2 = + 0\ 017$$

and therefore

$$\alpha_1 = 110°\ 34' \qquad h_1 = + 0\ 552$$
$$\alpha_2 = 149°\ 32' \qquad h_2 = − 0\ 020$$

$$f(\varphi) = 0\ 72 + 0\ 552 \sin(110°\ 34' + \varphi) + 0\ 020 \sin(149°\ 32' + 2\varphi) +$$

Ex 2 In a micrometer microscope, to find the correction for periodic error of the micrometer screw to the readings of the graduated micrometer head

The necessary observations are made by measuring in succession on a graduated scale the distance corresponding to intervals which are aliquot parts of a complete revolution of the screw

Let $A =$ the value of the space on the scale

$\varphi =$ the division on the micrometer head read on at the initial reading

The correction to the first reading will be

$$h_1 \sin(\alpha_1 + \varphi) + h_2 \sin(\alpha_2 + 2\varphi) +$$

and the correction to the second reading

$$h_1 \sin(\alpha_1 + A + \varphi) + h_2 \sin(\alpha_2 + 2A + 2\varphi) +$$

Hence, since the correction to the observed value M of the scale distance is the difference of these corrections, we have

$$A = M + 2h_1 \cos(\alpha_1 + \frac{A}{2} + \varphi) \sin \frac{A}{2} + 2h_2 \cos(\alpha_2 + A + 2\varphi) \sin A +$$

Suppose now that the scale has been read on by n different parts of the screw, so that to the observed values $M_1, M_2, \qquad M_n$ of the graduated distance correspond the values of φ,

$$\varphi, \varphi + \Theta, \qquad \varphi + (n - 1) \Theta$$

the amount by which the screw is shifted each time being $\Theta = \dfrac{360°}{n}$

We should then have n equations of the form

$$A = M + 2h_1 \cos(\alpha_1 + \frac{A}{2} + \varphi + m\Theta) \sin \frac{A}{2}$$
$$+ 2h_2 \cos(\alpha_2 + A + 2\varphi + 2m\Theta) \sin A +$$

where m assumes all values from o to $n - 1$

If we take $\dfrac{[M]}{n}$ as an approximate value of A, and put

$$M - \frac{[M]}{n} = l$$

the n observations may be expressed by the general formula

$$x + 2h_1 \cos(\alpha_1 + \frac{A}{2} + \varphi + m\Theta) \sin \frac{A}{2}$$
$$+ 2h_2 \cos(\alpha_2 + A + 2\varphi + 2m\Theta) \sin A + \qquad = l_m$$

where m assumes all values from o to $n - 1$

Expanding the cosines, the solution may be completed as in Art 194

The screw of the filar micrometer of the 26-in refractor of the U S Naval Observatory was examined for periodic error by Prof Hall in 1880 (*Washington Observations*, 1877)

The mii rometer was placed under the Harkness dividing engine, and the distance corresponding to each $\frac{1}{10}$ of a revolution of the screw was measured by means of the micrometer belonging to the engine   The following are the means for each $\frac{1}{10}$ of a revolution

Microm	M	l
o o to o 1	d o 6221	+ 9
1 " 2	6229	+ 17
2 " 3	6182	− 30
3 " 4	6212	o
4 " 5	6201	− 11
5 " 6	6227	+ 15
6 " 7	6226	+ 14
7 " 8	6150	− 62
8 " 9	6189	− 23
o 9 " 1 o	o 6285	+ 73
Mean,	o 6212 = o″ 995	

Assuming these residuals to have a periodic form, required the correction to the reading of the head of the micrometer

The observation equations are

$$x + y_1 \cos o \quad + z_1 \sin o \quad + y_2 \cos 2\ o + z_2 \sin 2\ o = \quad 9$$
$$x + y_1 \cos 36° + z_1 \sin 36° + y_2 \cos 72° + z_2 \sin 72° = 17$$

$$x + y_1 \cos 324° + z_1 \sin 324° + y_2 \cos 648° + z_2 \sin 648° = 73$$

The values of y_1, z_1,   are at once found from (a)
    The final result is

$$f(\varphi) = + 0'' 0002 \sin \varphi + 0'' 0022 \cos \varphi - 0'' 0022 \sin 2\varphi + 0'' 0047 \cos 2\varphi$$

Ex 3 To find the correction for periodic error of graduation to the value of an angle measured with a theodolite in which the circle is read by two opposite microscopes, 1 and 2

Let A = the value of the angle
    φ = the reading of the circle with microscope 1 when the telescope is pointed at the first signal

The correction to the reading of microscope 1 for periodic error will be

$$h_1 \sin(\alpha_1 + \varphi) + h_2 \sin(\alpha_2 + 2\varphi) +$$

and to the reading of microscope 2, writing $180 + \varphi$ for φ,

$$- h_1 \sin(\alpha_1 + \varphi) + h_2 \sin(\alpha_2 + 2\varphi) - \quad .$$

The correction to the mean reading on the first signal is, therefore,

$$h_2 \sin(\alpha_2 + 2\varphi) + h_4 \sin(\alpha_4 + 4\varphi) +$$

and the correction to the mean reading on the second signal,

$$h_2 \sin\left\{\alpha_2 + 2(A + \varphi)\right\} + h_4 \sin\left\{\alpha_4 + 4(A + \varphi)\right\} +$$

Hence, since the correction to the observed value M_1 of the angle is the difference of these corrections, we have

$$A = M_1 + 2h_2 \cos(\alpha_2 + A + 2\varphi)\sin A + 2h_4 \cos(\alpha_4 + 2A + 4\varphi)\sin 2A +$$

Suppose now, sighting at the same two signals, that readings have been made at n different parts of the circle, so that to the observed values of the angle $M_1, M_2, \quad M_n$ correspond the values of $\varphi, \varphi, \varphi + \Theta, \quad \varphi + (n-1)\Theta$, the amount by which the circle is shifted each time being Θ Complete the solution as in Ex 2

Given the measures of the angle Falkirk–Gasport–Pekin made with a Troughton and Simms 12 in theodolite

	M	l
	97° 22′ 36″ 80	− 1″ 03
	33″ 54	+ 2″ 23
	31″ 75	+ 4″ 02
	34″ 06	⊣ 1″ 71
	38″ 67	− 2″ 90
	39″ 63	− 4″ 06
Mean,	97° 22′ 35″ 77	

$$A = 97° \quad 22′ \quad 35″ \, 77$$
$$\varphi = 70° \quad 27′$$
$$n = 6, \quad \Theta = 60°$$

The observation equations,

$$y_2 \cos 0 \quad + z_2 \sin 0 \quad + y_4 \cos 0 \quad + z_4 \sin 0 \quad = + 1\ 03$$
$$y_2 \cos 60° + z_2 \sin 60° + y_4 \cos 120° + z_4 \sin 120° = - 2\ 23$$

$$y_2 \cos 300° + z_2 \sin 300° + y_4 \cos 600° + z_4 \sin 600° = + 4\ 06$$

The normal equations,

$$3y_2 \qquad\qquad = \quad 4\ 22$$
$$3z_2 \qquad = - 11\ 44$$
$$3y_4 \quad = - \ 1\ 04$$
$$3z_4 = \quad 0\ 55$$

Hence the correction for periodic error of any angle A measured with this instrument is given by

$$4'' \ 10 \cos (11° + A + 2\varphi) \sin A - 1'' \ 54 \cos (91° + 2A + 4\varphi) \sin 2A$$

197 Consult Bessel, *Abhandlungen*, vol 11 p 364, Chauvenet, *Astronomy*, vol 11 , Brunnow, *Astronomy*, Cole, *Great Trig Survey of India*, vol 11 , Woodward, *Report Chief of Engineers U S A* , 1876, 1879
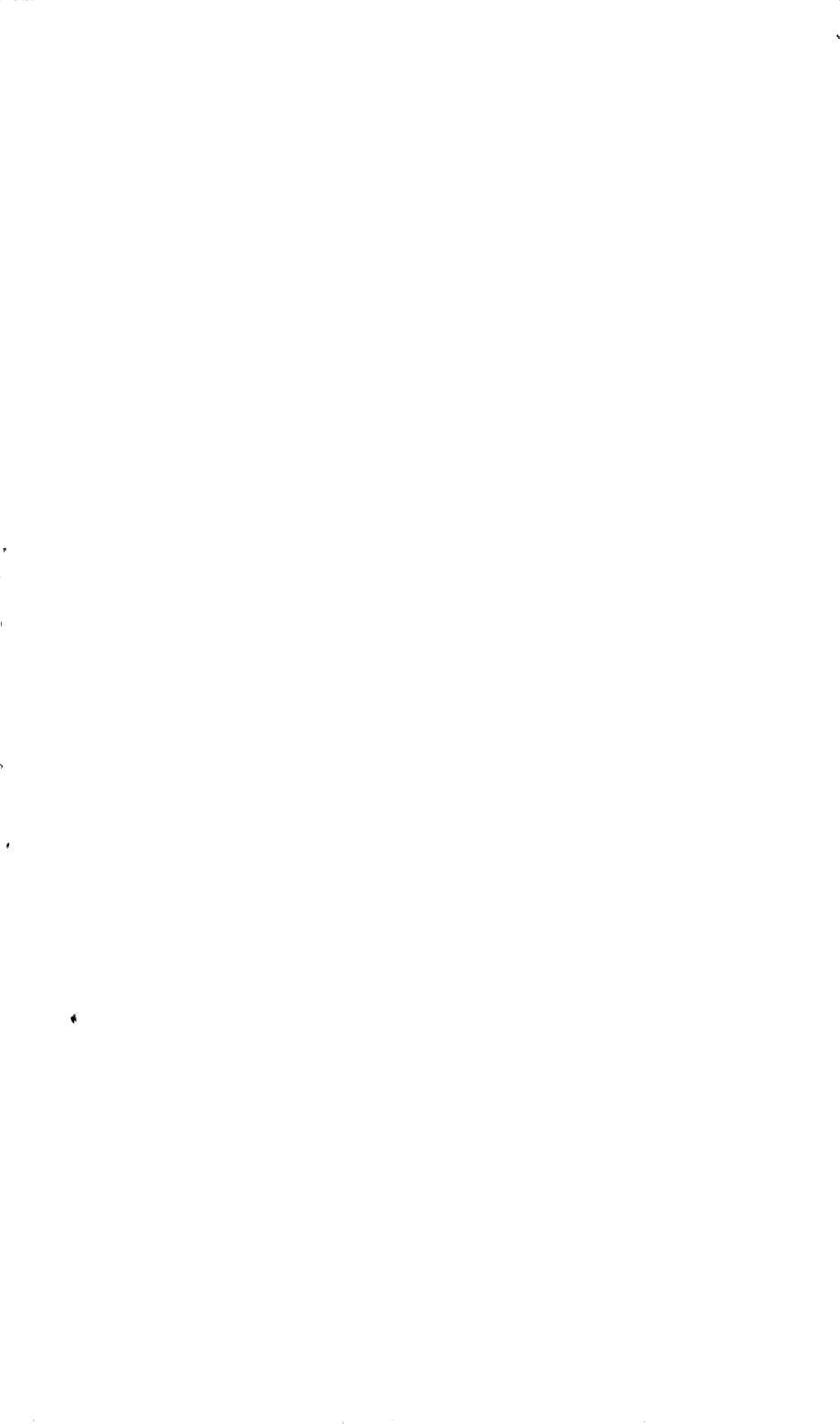
198 The first account of the method of least squares as now employed was published by Legendre in 1805, in his *Nouvelles méthodes pour la détermination des orbites des comètes*. The main points on which he insists are the generality and ease of application of the method He states without proof the principle that the sum of the squares of the errors must be made a minimum, and deduces the arithmetic mean as a special case of this general statement He was also the first to make use of the term least squares (*moindres quarrés*).

The first demonstration of the exponential law of error was published in 1808 by Dr R. Adrain, of Reading, Pa., in the *Analyst* or *Mathematical Museum*.* He gives two demonstrations, the second being essentially the same as that now known as Herschel's proof After deriving the principle of minimum squares Adrain applied it to the solution of the following four problems:

(1) Supposing $a, b, c, \ldots$ to be the observed measures of any quantity x, the most probable value of x is required.

(2) Given the observed positions of a point in space, to find the most probable position of the point.

(3) To correct the dead reckoning at sea by an observation of the latitude.

(4) To correct a survey. (Ex. 5, Art. 110.)

Gauss published in 1809, in the *Theoria motus corporum cælestium*, a third proof of the law of error, thence deducing the principle of minimum squares. He so thoroughly de-

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* There is a copy of the *Analyst* containing Adrain's proofs in the library of the American Philosophical Society Philadelphia An account of Adrain's life will be found in the *Democratic Review* for 1844

veloped the subject in its principles and practical applica
tions that comparatively little has been added by later
writers. An English translation of the *Theoria motus* by
Admiral Davis, U S N., was published in 1858, and a
French translation of Gauss' memoirs on least squares by
Bertrand in 1855.

The main contributions to the subject, aside from those
mentioned, have been made by Laplace in the fundamental
principles, and by Bessel, Hansen, and Andræ in its applica
tions to astronomical and geodetic work.

199. Many proofs of the law of error have been given.
The most important are by Adrain, Gauss, Laplace, and
Hagen.

That by Gauss in 1809 assumes that if a series of obser
vations, all equally good, are made of a quantity, the arith
metic mean of the observed values is the most probable
value of the quantity. The law of error is then deduced,
and the principle of minimum squares follows at once.

Laplace, in his first proof (1810), deduces the principle
that the sum of the squares of the errors must be made a
minimum without reference to any law of error, only as
suming that positive and negative errors are equally prob
able and that the number of observations is infinitely great.
A clear account of this proof is given by Glaisher, *Mem.
Roy. Astron. Soc.*, vol. xxxix., and by Meyer, *Wahrschein
lichkeitsrechnung*, pp. 440–473. See also Airy, *Theory of
Errors of Observations*, second edition; Todhunter, *History of
the Theory of Probability*.

Hagen's proof (1837) is founded on the hypothesis that
an error of observation is the algebraic sum of an infinite
number of element errors which are all of equal value and
which are as likely to be positive as negative. A modified
form of this proof is given in Appendix II.

Adrain's second proof is shorter than any of the three
mentioned, but is not so satisfactory. This proof is also
given by Sir John Herschel, *Edinburgh Review*, vol. xcii.;
Airy, *Theory of Errors of Observations*, third edition; Thomson

and Tait, *Treatise on Natural Philosophy*, vol. i.; Clarke, *Geodesy*.

For more extended information on this subject the reader is referred to Merriman's very complete memoir entitled *List of Writings Relating to the Method of Least Squares*, New Haven, 1877

APPENDIX II

THE LAW OF ERROR

Proof on Hagen's [Young's] Hypothesis

200. If a quantity, T, is to be determined, and M_1 is an observed value of T, then, if the observation were perfect, we should have

$$T - M_1 = 0$$

But since, if we make a second and a third observation, we may not find the same value as we did at first, and as we can only account for the difference on the supposition that the observations are not perfect—that is, that they are affected with certain errors—we should rather write

$$T - M_1 = \Delta_1$$
$$T - M_2 = \Delta_2$$
$$T - M_3 = \Delta_3$$

where M_1, M_2, M_3 are the observed values, and Δ_1, Δ_2, Δ_3 are the errors of the observations.

Now, an error of observation does not result from a single cause. Thus in reading an angle with a theodolite the error in the value found is the result of imperfect adjustment of the instrument, of various atmospheric changes

of want of precision in the observer's method of handling
the instrument, etc   Each of these influences may be taken
as the result of numerous other influences   Thus the first
mentioned may include errors of collimation, of level, etc
Each of these in turn may be taken as resulting from other
influences, and so on. The final influences, or element
errors, as they may be called, must be assumed to be inde-
pendent of one another, and each as likely to make the
resultant error too large as too small   that is, as likely to
be positive as negative. The number of these element
errors being very great, we may, from the impossibility of
assigning the limit, consider it as infinite in any case. Each
element error must consequently be an infinitesimal, and
for greater simplicity we may take those occurring in any
one series as of the same numerical magnitude. Hence we
conclude that an error of observation may be assumed to
be the algebraic sum of a very great number of independent
infinitesimal element errors ι, all equal in magnitude, but as
likely to be positive as negative.

Let the number of these element errors be denoted by
$2n$, as the generality of the demonstration will not be affected
by supposing this infinitely great number to be even. If
all of the element errors are $+$, the error $2n\iota$ results, and
this can occur in but one way ; if all but one are $+$, the
error $(2n - 2)\iota$ results, and this can occur in $2n$ ways; and
generally if $n + m$ are $+$, and $n - m$ are $-$, the error $2m\iota$
results, and this can occur in $\dfrac{2n\,(2n - 1)\,.\,.\,.\,(n + m + 1)}{1\;2\,.\,.\,.\,(n - m)}$
ways.* Hence the numbers expressing the relative fre-
quency of the errors (that is, the number of times they may
be expected to occur) are equal to the coefficients in the
development of the $2n^{th}$ power of any binomial.

The element errors, infinite in number, being infinitely
small in comparison with the actual errors of observation,
these latter may consequently be assumed to be continuous
from o to $2n\iota$, the maximum error. If, therefore, d denotes

* See Todhunter's or Newcomb's *Algebra.*

the error in which $n + m$ $+ \varepsilon$'s and $n - m$ $- \varepsilon$'s occur, and $\Delta + d\Delta$ denotes the consecutive error in which $n + m + 1$ $+ \varepsilon$'s and $n - m - 1$ $- \varepsilon$'s occur, we have

$$\Delta = 2m\varepsilon$$
$$\Delta + d\Delta = (2m + 2)\varepsilon$$

and therefore
$$\Delta = md\Delta$$

Calling f the relative frequency of the error Δ, and $f + df$ that of the consecutive error $\Delta + d\Delta$, we have

$$f = \frac{2n(2n - 1) \qquad (n + m + 1)}{n - m!}$$

$$f + df = \frac{2n(2n - 1) \qquad (n + m + 2)}{n - m - 1!}$$

Hence, by division,

$$\frac{f + df}{f} = \frac{n - m}{n + m + 1}$$

or

$$\frac{df}{f} = -\frac{2m + 1}{n + m + 1}$$

$$= -\frac{2\Delta + d\Delta}{nd\Delta + \Delta + d\Delta}$$

Now, since $d\Delta$ is infinitely small in comparison with Δ, we may write
$$\frac{df}{f} = -\frac{2\Delta}{nd\Delta + \Delta}$$

Also, since df is infinitely small in comparison with f, 2Δ is with respect to $nd\Delta + \Delta$, and we may neglect Δ in the denominator in comparison with $nd\Delta$. We have, therefore,

$$\frac{df}{f} = -\frac{2\Delta}{nd\Delta}$$

And since d is infinitely small in comparison with $x\,dd$, and dd is infinitely small in comparison with d, it follows that x must be an infinity of the second order. It is, therefore, of a magnitude comparable with $\frac{1}{(dd)^2}$, and hence $x(dd)^2$ must be a finite constant. Calling this constant $\frac{1}{h^2}$, we have

$$\frac{df}{f} = -2h^2\,d\,dd$$

Integrating and denoting the value of f, when $d = 0$, by f_0,

$$f = f_0 e^{-h^2 d^2}$$

The errors being separated by the intervals dd, so that $0, dd, \ldots d, d + dd, \ldots$ are the errors in order of magnitude, we must, in order to make the system consistent with the definition of probability, and therefore continuous, consider not so much the relative frequency of the detached errors as the relative frequency of the errors within certain limits.

Now, by the definition of probability, the probability of an error between the limits d and $d + dd$ is represented by a fraction whose numerator is the number of errors which fall between d and $d + dd$, and denominator the total number of errors committed. If we denote this probability by $\varphi(d)$ we may write

$$\varphi(d) = \frac{f}{\Sigma f}$$

$$= c\,e^{-h^2 d^2}$$

where c is a constant, Σf being necessarily a constant for the same series of observations.

201. The principle of least squares now follows readily.

Suppose that a series of observations has been made of a function T of a certain known form,

$$T = f(X, Y, \ldots)$$

in which the constants that enter are given by theory for each observation, and X, Y, . are the unknowns to be found

Call M_1, M_2, . the observed values of T, and T_1, T_2, the corresponding true values of the function, so that the errors are found from

$$T_1 - M_1 = \Delta_1$$
$$T_2 - M_2 = \Delta_2$$
$$. \quad . \quad . \quad .$$

If we knew the true values of X, Y, , and therefore of T, we should have for the probabilities, before the first, second, observations are made, that the errors to be expected lie between Δ_1 and $\Delta_1 + d\Delta_1$, Δ_2 and $\Delta_2 + d\Delta_2$, respectively, are

$$c_1 e^{-h_1^2\Delta_1^2}, c_2 e^{-h_2^2\Delta_2^2}, \quad .$$

where c_1, c_2, h_1, h_2, are constants

The probability ψ of the simultaneous occurrence of all of these errors, which are independent of each other, is given by (Art 5)

$$\psi = c_1 c_2, \quad e^{-[h^2\Delta^2]}$$

But as only the observed values M_1, M_2, are known, the true values of T, X, Y, , Δ_1, Δ_2, , and therefore of ψ, are unknown

If now arbitrary values of X, Y. . are assumed, T, Δ_1, Δ_2, will receive values, and therefore a value of ψ will be determined Other assumed values of X, Y, . will give other values of ψ, which is therefore a function of X, Y, Of all possible values which are given to X, Y, there must be some one set of values which is to be chosen in preference to any others The most probable set is naturally the one that would be chosen

Let, then, X_1, Y_1, . denote the most probable values of X, Y, Substitute them in the function and let

$V_1, V_2,$ denote the resulting values of $T_1, T_2,$. Then we have no longer the true errors $T_1 - M_1, T_2 - M_2,$ ·, but the errors $V_1 - M_1, V_2 - M_2, \cdot \cdot$, which may be called *residual errors* of observation, being the difference between the most probable and the observed values. They are denoted by the symbols $v_1, v_2,$.

Now, assuming that $c\, e^{-h^2v^2}$ denotes the probability of a residual between v and $v + dv$, the expression for ψ becomes

$$\psi = c_1 c_2 \cdot \quad \cdot \, e^{-[h^2v^2]}$$

and the most probable set of values of $X, Y, \cdot \cdot$ would be that which corresponds to the maximum value of this expression, which can only happen when

$$[h^2v^2] \text{ is a minimum}$$

for the same set of values of $X, Y,$

· This is the principle of least squares

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TABLE I

Values of $\Theta(t) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_{0}^{\rho \frac{a}{r}} e^{-t^{2}} dt$

$\dfrac{a}{r}$	$\Theta(t)$	Diff	$\dfrac{a}{r}$	$\Theta(t)$	Diff
0,0	0,000	54	2,5	0,908	13
0,1	0,054	53	2,6	0,921	10
0,2	0,107	53	2,7	0,931	10
0,3	0,160	53	2,8	0,941	9
0,4	0,213	51	2,9	0,950	7
0,5	0,264	50	3,0	0,957	6
0,6	0,314	49	3,1	0,963	6
0,7	0,363	48	3,2	0,969	5
0,8	0,411	45	3,3	0,974	4
0,9	0,456	44	3,4	0,978	4
1,0	0,500	42	3,5	0,982	3
1,1	0,542	40	3,6	0,985	2
1,2	0,582	37	3,7	0,987	3
1,3	0,619	36	3,8	0,990	1
1,4	0,655	33	3,9	0,991	2
1,5	0,688	31	4,0	0,993	1
1,6	0,719	29	4,1	0,994	1
1,7	0,748	27	4,2	0,995	1
1,8	0,775	25	4,3	0,996	1
1,9	0,800	23	4,4	0,997	1
2,0	0,823	20	4,5	0,998	0
2,1	0,843	19	4,6	0,998	0
2,2	0,862	17	4,7	0,998	1
2,3	0,879	16	4,8	0,999	0
2,4	0,895	13	4,9	0,999	0
2,5	0,908		5,0	0,999	

APPENDIX

TABLE II

Factors for Bessel's Probable-Error Formulas

n	$\dfrac{6745}{\sqrt{n-1}}$	$\dfrac{6745}{\sqrt{n(n-1)}}$	n	$\dfrac{6745}{\sqrt{n-1}}$	$\dfrac{6745}{\sqrt{n(n-1)}}$
			40	0 1080	0 0171
			41	1066	0167
2	0 6745	0 4769	42	1053	0163
3	4769	2754	43	1041	0159
4	3894	1947	44	1029	0155
5	0 3372	0 1508	45	0 1017	0 0152
6	3016	1231	46	1005	0148
7	2754	1041	47	0994	0145
8	2549	0901	48	0984	0142
9	2385	0795	49	0974	0139
10	0 2248	0 0711	50	0 0964	0 0136
11	2133	0643	51	0954	0134
12	2029	0587	52	0944	0131
13	1947	0540	53	0935	0128
14	1871	0500	54	0926	0126
15	0 1803	0 0465	55	0 0918	0 0124
16	1742	0435	56	0909	0122
17	1686	0409	57	0901	0119
18	1636	0386	58	0893	0117
19	1590	0365	59	0886	0115
20	0 1547	0 0346	60	0 0878	0 0113
21	1508	0329	61	0871	0111
22	1472	0314	62	0864	0110
23	1438	0300	63	0857	0108
24	1406	0287	64	0850	0106
25	0 1377	0 0275	65	0 0843	0 0105
26	1349	0265	66	0837	0103
27	1323	0255	67	0830	0101
28	1298	0245	68	0824	0100
29	1275	0237	69	0818	0098
30	0 1252	0 0229	70	0 0812	0 0097
31	1231	0221	71	0806	0096
32	1211	0214	72	0800	0094
33	1192	0208	73	0795	0093
34	1174	0201	74	0789	0092
35	0 1157	0 0196	75	0 0784	0 0091
36	1140	0190	80	0759	0085
37	1124	0185	85	0736	0080
38	1109	0180	90	0713	0075
39	1094	0175	100	0678	0068

TABLE III

Factors for Peters' Probable Error Formulas

n	$0.4541 \\ \dfrac{1}{\sqrt{n(n-1)}}$	$0.4541 \\ \dfrac{1}{n\sqrt{n-1}}$	n	$0.4541 \\ \dfrac{1}{\sqrt{n(n-1)}}$	$0.4541 \\ \dfrac{1}{n\sqrt{n-1}}$
			40	0.0214	0.0034
			41	0.0209	0.0033
2	0.5978	0.4227	42	0.0204	0.0031
3	0.3451	0.1993	43	0.0199	0.0030
4	0.2440	0.1220	44	0.0194	0.0029
5	0.1930	0.0845	45	0.0190	0.0028
6	0.1620	0.0660	46	0.0186	0.0027
7	0.1418	0.0535	47	0.0182	0.0027
8	0.1272	0.0450	48	0.0178	0.0026
9	0.1157	0.0386	49	0.0174	0.0025
10	0.1062	0.0336	50	0.0171	0.0024
11	0.0984	0.0297	51	0.0167	0.0023
12	0.0917	0.0265	52	0.0164	0.0023
13	0.0860	0.0238	53	0.0161	0.0022
14	0.0810	0.0217	54	0.0158	0.0022
15	0.0767	0.0198	55	0.0155	0.0021
16	0.0728	0.0182	56	0.0152	0.0020
17	0.0693	0.0168	57	0.0150	0.0020
18	0.0661	0.0156	58	0.0147	0.0019
19	0.0632	0.0145	59	0.0145	0.0019
20	0.0606	0.0135	60	0.0143	0.0018
21	0.0582	0.0127	61	0.0140	0.0018
22	0.0560	0.0119	62	0.0137	0.0017
23	0.0539	0.0112	63	0.0135	0.0017
24	0.0520	0.0106	64	0.0133	0.0017
25	0.0503	0.0100	65	0.0131	0.0016
26	0.0487	0.0095	66	0.0129	0.0016
27	0.0472	0.0090	67	0.0127	0.0016
28	0.0457	0.0086	68	0.0125	0.0015
29	0.0443	0.0082	69	0.0123	0.0015
30	0.0431	0.0079	70	0.0122	0.0015
31	0.0419	0.0075	71	0.0120	0.0014
32	0.0408	0.0072	72	0.0118	0.0014
33	0.0397	0.0069	73	0.0117	0.0014
34	0.0387	0.0066	74	0.0115	0.0013
35	0.0378	0.0064	75	0.0113	0.0013
36	0.0369	0.0062	80	0.0106	0.0012
37	0.0360	0.0059	85	0.0100	0.0011
38	0.0352	0.0057	90	0.0095	0.0010
39	0.0344	0.0055	100	0.0085	0.0009